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CHARACTERIZATION OF CHAOTIC ORDER AND ITS APPLICATION TO FURUTA INEQUALITY

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Dedicated to Professor P. R. Halmos on his 80th Birthday

Abstract. In this note, we give a simple characterization of the chaotic order $\log A \geq \log B$ among positive invertible operators A, B on a Hilbert space. As an application, we discuss Furuta's type operator inequality.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is positive, in symbols $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. And $A > 0$ means that A is positive invertible. It is well-known that $A \geq B \geq 0$ does not assure $A^2 \geq B^2$ in general, but the Löwner-Heinz inequality says that the function $t \to t^\alpha$ on $[0, \infty)$ is operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha, \tag{1}$$

cf. [8]. Furuta [5] gave it an ingenious extension which is called the Furuta inequality (cf. [2], [7] and [6] for an elementary and one-page proof):

If $A \geq B \geq 0$, then

$$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q} \tag{2}$$

and

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} \tag{3}$$

holds for $r \geq 0,\ p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$ (see Figure 1).

Since $\log\ t$ is operator monotone, i.e., $\log\ A \geq \log\ B$ for $A \geq B > 0$, it induces a weaker order $\gg$ among positive invertible operators than the usual one $\geq$, which is called the chaotic order, cf. [3]. Now Ando's theorem [1] is rephrased as a characterization of the chaotic order via a form of (2): For $A, B > 0$, $A \gg B$ if and only if

$$(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} \leq A^p \tag{4}$$

holds for all $p \geq 0$.

Afterwards, it is extended to the following result [3], cf. [4].

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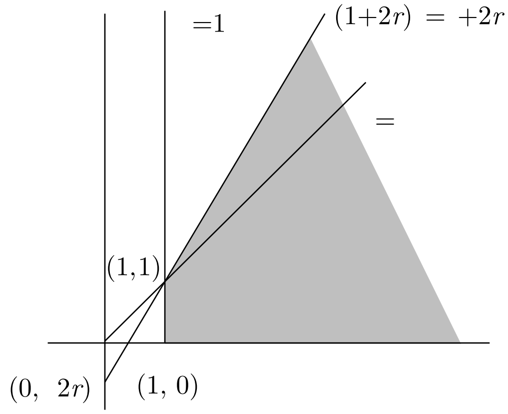

FIGURE 1

Theorem A. *For $A, B > 0$, $A \gg B$ if and only if*

$$(A^r B^p A^r)^{\frac{2r}{p+2r}} \leq A^{2r} \tag{5}$$

holds for all $p, r \geq 0$.

In this note, we give a simple characterization of the chaotic order. Precisely, $\log\ A > \log\ B$ if and only if there is an $\alpha > 0$ such that

$$A^\alpha > B^\alpha. \tag{6}$$

As an application, we can obtain Furuta's type operator inequality implying Theorem A.

2. Characterization of chaotic order

We begin by stating a simple lemma which is the heart of this note:

Lemma 1. *If A and B are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that*

$$e^{\alpha A} > e^{\alpha B}. \tag{7}$$

Proof. The assumption $A > B$ means that $A - B \geq \varepsilon > 0$ for some ε. We here take $0 < \alpha < \varepsilon/(e^{\|A\|} + e^{\|B\|})$ and $\alpha \leq 1$. Then we have

$$\begin{aligned} e^{\alpha A} - e^{\alpha B} &= \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!}(A^n - B^n) \\ &\geq \alpha\varepsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \\ &\geq \alpha\varepsilon - \alpha^2 \| \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \| \\ &\geq \alpha\varepsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!}(\|A\|^n + \|B\|^n) \\ &\geq \alpha(\varepsilon - \alpha(e^{\|A\|} + e^{\|B\|})) > 0. \end{aligned}$$

□

Lemma 1 implies the following basic inequality:

Corollary 2. *If* $A, B > 0$, *then* $\log\ A > \log\ B$ *if and only if there exists an* $\alpha \in (0,1]$ *such that* $A^\alpha > B^\alpha$.

Proof. If $\log\ A > \log\ B$, then $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$ by Lemma 1. Conversely, if $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$, then $A^\alpha \geq B^\alpha + \delta$ for some $\delta > 0$ and

$$\alpha \log A = \log A^\alpha \geq \log(B^\alpha + \delta) > \log B^\alpha = \alpha \log B.$$

□

By the above discussion, we have the following simple characterization of the chaotic order:

Theorem 3. *For* $A, B > 0$, $A \gg B$, *i.e.,* $\log\ A \geq \log\ B$, *if and only if for any* $\delta \in (0,1]$ *there exists an* $\alpha = \alpha_\delta > 0$ *such that*

$$(e^\delta A)^\alpha > B^\alpha. \tag{8}$$

Proof. Since $A \gg B$ is equivalent to $\log e^\delta A = \log A + \delta > \log B$ for any $\delta > 0$, Corollary 2 implies that $A \gg B$ is equivalent to saying that for any $\delta > 0$ there exists an $\alpha = \alpha_\delta \in (0,1]$ such that $(e^\delta A)^\alpha > B^\alpha$. □

We comment that some inequalities related to the chaotic order can be obtained from our result. Among others, we here discuss the Furuta inequality under the chaotic order. Combining Theorem 3 and the Furuta inequality, we have the following lemma:

Lemma 4. *If* $A, B > 0$ *and* $A \gg B$, *then for any* $\delta > 0$ *there exists an* $\alpha = \alpha_\delta \in (0,1]$ *such that*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}} \tag{9}$$

holds for $p \geq 0$, $r \geq 0$ *and* $q \geq 1$ *with* $(\alpha + 2r)q \geq p + 2r$.

Thus we have the following result equivalent to Theorem A:

Theorem 5. *If* $A, B > 0$ *and* $A \gg B$, *then*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}} \tag{10}$$

holds for $p \geq 0$, $r \geq 0$ *and* $q \geq 1$ *with* $2rq \geq p + 2r$.

The point of the proof is that if p, q and r satisfy the above condition, then $(\alpha + 2r)q \geq p + 2r$ for all $\alpha > 0$.

References

1. T.Ando, *On some operator inequalities*, Math. Ann., **279** (1987), 157-159. MR **89c:**47019
2. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, **23** (1990), 67-72. MR **91g:**47012
3. M.Fujii,T.Furuta and E.Kamei, *Operator functions associated with Furuta's inequality*, Linear Alg. and its Appl., **179** (1993), 161-169. MR **93j:**47026
4. M.Fujii and E.Kamei, *Furuta's inequality and a generalization of Ando's Theorem*, Proc. Amer. Math. Soc., **115** (1992), 409-413. MR **92i:**47018
5. T.Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0, p \geq 0, q \geq 1$ *with* $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., **101** (1987), 85-88. MR **89b:**47028
6. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g:**47029

7. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883-886. MR **89m:**47011
8. G.K.Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc., **36** (1972), 309–310. MR **46:**6078

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