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FACTORIZATION OF AN INTEGRALLY CLOSED IDEAL IN TWO-DIMENSIONAL REGULAR LOCAL RINGS

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ABSTRACT. Let (R, m, k) be a two-dimensional regular local ring with algebraically closed residue field k and I be an m-primary integrally closed ideal in R. Let $T(I)$ be the set of Rees valuations of I and $k(v)$ be the residue field of the valuation ring V associated with $v \in T(I)$. Assume that (a, b) is any minimal reduction of I. We show that if I is the product of the distinct simple m-primary integrally closed ideals in (R, m, k), then $k(v)$ is generated by the image of a/b over k for all $v \in T(I)$, and the converse of this is also true.

1. Introduction

Throughout this paper (R, m, k) will denote a 2-dimensional regular local ring (**RLR** for short) with residue field k and quotient field K. Let I be an m-primary integrally closed ideal in (R, m, k). Concerning the structure of the integrally closed ideals in a 2-dimensional **RLR** (R, m, k), O. Zariski proved three beautiful theorems which are the main background for this paper. Zariski's Product Theorem ([8], Appendix 5, Theorem 2′) says that any product of integrally closed ideals is integrally closed. Hence the set of Rees valuations of I is

$$T(I) = \bigcup_{\substack{a \neq 0 \\ a \in I}} \left\{ v \mid v \text{ is the valution of } \big(R[I/a]\big)_q,\ q \in \operatorname{Min}\big(aR[I/a]\big) \right\}.$$

By Zariski's Unique Factorization Theorem ([8], Appendix 5, Theorem 3), $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$, $\mu_i \geq 1$, where $I_1, \cdots, I_l$ are distinct simple m-primary integrally closed ideals in (R, m, k). Zariski also set up an one-to-one correspondence between the set of simple integrally closed ideals of R and the set of prime divisors of the second kind on R ([8], Appendix 5, Theorem E). Therefore, if $\operatorname{Min}(mR[I/a]) = \{ q_1, \cdots, q_\lambda \}$ then $\lambda = l$ and, upon reordering, v_i is the valuation of $(R[I/a])_{q_i}$, where v_i is the prime divisor associated to the ideal I_i for $i = 1, \cdots, l$.

We denote by $k(v)$ the residue field of the valuation ring V associated with v. Assume that k is an algebraically closed field. Then $k(v)$ is a simple transcendental field extension of k for all $v \in T(I)$. Moreover, if (a, b) is any minimal reduction of I, then for all $v \in T(I)$, the image of a/b in $k(v)$ is transcendental over k. We show that if $I = I_1 \cdots I_l$, where $I_1, \cdots, I_l$ are distinct simple m-primary integrally closed ideals in (R, m, k), then $k(v)$ is generated by the image of a/b over k for

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all $v \in T(I)$, and the converse of this is also true. Section 2 is devoted to some preliminaries. In section 3, we will prove main results.

2. Preliminaries

Let (A, n) be a local ring and I an ideal of A. An ideal J contained in I is called a reduction of I if $JI^s = I^{s+1}$ for some integer $s \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I. The order $o(I)$ of an ideal I of a local ring (A, n) is r if $I \subseteq n^r$ but $I \not\subseteq n^{r+1}$. We will use notation $e(I)$ to denote the multiplicity of an n-primary ideal I of A. Recall that an ideal is simple if it is not the unit ideal and has no nontrivial factorization. An element $a \in A$ is said to be integral over an ideal I of A if a satisfies an equation of the form

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0, \qquad r_i \in I^i.$$

The set of all elements in A which are integral over an ideal I forms an ideal, denoted by $\overline{I}$ and called the integral closure of I. An ideal I is said to be integrally closed (or equivalently "complete") if $I = \overline{I}$.

In a d-dimensional local domain (A, n, l) with quotient field L, by a prime divisor of the second kind on A (or equivalently prime divisor of (A, n)) we mean a discrete valuation v of L on A which is non-negative on A and has center n on A and whose residual transcendence degree (denoted by $\text{tr.deg}_l k(v)$) is $d - 1$.

Lemma 2.1. *Let (A, n) be a quasilocal normal domain with quotient field L. If $u \in L \backslash A$ is such that $u^{-1} \notin A$, then $nA[u]$ is a prime ideal in $A[u]$ and $A[u]/nA[u] \cong (A/n)[X]$, a polynomial ring in one-variable over the field A/n.*

Proof. Define the canonical homomorphism ϕ from $A[X]$ onto $A[u]$ with $\phi(X) = u$. $\text{Ker}(\phi)$ is a prime ideal in $A[X]$ since $A[u]$ is a domain. By Theorem 11.13. in [7], $\text{Ker}(\phi)$ is generated by linear polynomials $cX - d$ with $u = d/c$, where $c, d \in A$. It is not difficult to see that c and d are in the maximal idael n of A. Hence we have the following exact sequence :

$$0 \longrightarrow \frac{nA[X]}{\text{Ker}(\phi)} \longrightarrow \frac{A[X]}{Ker(\phi)} \xrightarrow{\overline{\phi}} \frac{A[u]}{nA[u]} \longrightarrow 0,$$

where $\overline{\phi}$ is the map induced by ϕ. Hence we have $A[u]/nA[u] \cong A[X]/nA[X]$ is a polynomial ring one-variable over the field A/n, and so $nA[u]$ is a prime ideal in $A[u]$. □

3. Main results

The following lemma will play a key role in the proofs of the first main result.

Lemma 3.1. *Let I be an m-primary integrally closed ideal in (R, m, k). Assume that $Min(mR[It]) = \{p_1, \cdots, p_l\}$ and that k is an algebraically closed field. Let v_i be the valuation of $R[It]_{p_i} \cap K$ for $i = 1, \cdots, l$. Then the following conditions are equivalent.*

1. *$R[It]/p_i$ is regular for $i = 1, \cdots, l$.*
2. *$I = I_1 \cdots I_l$, where $I_1, \cdots, I_l$ are distinct simple m-primary integrally closed ideals in R.*

(3) *For any reduction* (a,b) *of* I*, there exist elements* $c_{i_1},\cdots,c_{i_n}$ *in* I *such that* $a,b,c_{i_1},\cdots,c_{i_n}$ *is a minimal generating set of* I *and* $v_i(c_{i_j}) > v_i(a) = v_i(b)$ *for* $i = 1,\cdots,l$ *and* $j = 1,\cdots,n$.

(4) *There exist a reduction* (a,b) *of* I *and elements* $c_{i_1},\cdots,c_{i_n}$ *in* I *such that* $a,b,c_{i_1},\cdots,c_{i_n}$ *is a minimal generating set of* I *and* $v_i(c_{i_j}) > v_i(a) = v_i(b)$ *for* $i = 1,\cdots,l$ *and* $j = 1,\cdots,n$.

Proof. (1)$\Longleftrightarrow$ (2) See ([5], Theorem 3.1).
(1)$\Longleftrightarrow$ (3)$\Longleftrightarrow$(4) See ([5], Theorem 3.3). □

Theorem 3.2. *Let* I *be an* m*-primary integrally closed ideal in* (R,m,k) *with an algebraically closed field* k *and* (a,b) *be any minimal reduction of* I*. Assume that* $I = I_1\cdots I_l$*, where* $I_1,\cdots,I_l$ *are distinct* m*-primary simple integrally closed ideals in* R*. Then* $k(v)$ *is generated by the image of* a/b *over* k *for all* $v \in T(I)$.

Proof. Suppose that $I = I_1\cdots I_l$, where $I_1,\cdots,I_l$ are distinct simple m-primary integrally closed ideals in R. Let v_i be a prime divisor of R associated to I_i for $i = 1,\cdots,l$. Let $\mathrm{Min}(mR[I/b]) = \{q_1,\cdots,q_\lambda\}$. By Zariski's One-to-One Correspondence Theorem, we have that $\lambda = l$ and, upon reordering, v_i is the discrete valuation of $R[I/b]_{q_i}$ for $i = 1,\cdots,l$, *i.e.*, $T(I) = \{v_1,\cdots,v_l\}$. By Lemma 3.1, there exist elements $c_{i_1},\cdots,c_{i_n}$ in I such that $a,b,c_{i_1},\cdots,c_{i_n}$ is a minimal generating set of I and $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $i = 1,\cdots,l$ and $j = 1,\cdots,n$. For each $i = 1,\cdots,l$, let $J_i = (m, c_{i_1}/b,\cdots,c_{i_n}/b)$. Then $J_i \subseteq q_i$ since $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $j = 1,\cdots,n$. And we have

$$\begin{aligned} R[I/b]/J_i &= R[a/b]/mR[a/b] \\ &\cong (R/m)[X] \qquad \text{by Lemma 3.1.} \end{aligned}$$

Let $(a/b)^*_i$ be the image of a/b in $R[I/b]/J_i$ for $i = 1,\cdots,l$. Since J_i is a prime ideal in $R[I/b]$ and $\dim(R[I/b]/J_i) = \dim(R[I/b]/q_i) = 1$, we have $J_i = q_i$ for $i = 1,\cdots,l$. Thus $R[I/b]/q_i = (R/m)[(a/b)^*_i]$ for $i = 1,\cdots,l$, which is a polynomial ring in one-variable over k. Localizing at q_i, we have $k(v_i) = (R/m)((a/b)^*_i)$ for $i = 1,\cdots,l$. □

Corollary 3.3 ([3], Remark 3.5). *Let* I *be an* m*-primary integrally closed ideal in* (R,m,k) *with an algebraically closed field* k *and* (a,b) *be any minimal reduction of* I*. Assume that* I *is simple. Let* v *be a prime divisor of* R *associated to* I*. Then* $k(v)$ *is generated by the image of* a/b *over* k.

We remark that Theorem 3.2 does not extend, in general, to the case where I is a power of a simple ideal.

Example 3.4. Let $R = k[x,y]_{(x,y)}$ and $m = (x,y)R$ with an algebraically closed field k. Let $I = m^2$. Then $T(I) = \{o\}$, where o is the order valuation, *i.e.*, the Rees valuation of I is the m-adic prime divisor of R, $R[x/y]_{mR[x/y]}$. (x^2,y^2) is a minimal reduction of I. Since $m = (x,y)$ and $T(m) = \{o\}$, by Theorem 3.2, $k(o) = k(\theta)$, where θ is the image of x/y in $R[x/y]/mR[x/y]$. Since $o(x^2) = o(y^2) = 2$, we have that x^2/y^2 is unit in $R[x/y]_{mR[x/y]}$, and hence the image of x^2/y^2 in $R[x/y]/mR[x/y]$ is θ^2. But $k(\theta^2) \subsetneq k(\theta)$.

Lemma 3.5 (Lipman, [2]). *Let* I *be an* m*-primary integrally closed ideal in* (R,m,k) *with an algebraically closed field* k*. Assume that* $I = I_1^{\mu_1}\cdots I_l^{\mu_l}$ *is the* *unique factorization of I as a product simple integrally closed ideals $I_1, \cdots, I_l$. Then*

$$e(I) = \sum_{i=1}^{l} \mu_i v_i(I),$$

where v_i is the prime divisor associated to I_i for $i = 1, \cdots, l$.

Lemma 3.6 ([4], Theorem 1.1). *Let (A, n) be a local ring with infinite residue field and $J = (x_1, \cdots, x_d)A$ an ideal generated by a system of parameters of A. Set*

$$T = A[x_1/x_d, \cdots, x_{d-1}/x_d]_{nA[x_1/x_d, \cdots, x_{d-1}/x_d]}.$$

Then $e(J) = e(JT)$.

Theorem 3.7. *Let I be an m-primary integrally closed ideal in (R, m, k) with an algebraically closed field k. Assume that $T(I) = \{v_1, \cdots, v_l\}$. Then the following conditions are equivalent.*

(1) *$I = I_1 \cdots I_l$, where $I_1, \cdots, I_l$ are distinct simple m-primary integrally closed ideals in R.*
(2) *For any reduction (a, b) of I, $k(v_i) = k((a/b)_i^*)$, where $(a/b)_i^*$ is the image of a/b in $k(v_i)$ for $i = 1, \cdots, l$.*
(3) *For some reduction (a, b) of I, $k(v_i) = k((a/b)_i^*)$, where $(a/b)_i^*$ is the image of a/b in $k(v_i)$ for $i = 1, \cdots, l$.*

Proof. (1)$\Rightarrow$ (2) : This follows immediately from Theorem 3.2.

(2)$\Rightarrow$ (3) : It is clear.

(3)$\Rightarrow$ (1) : Assume that (3) is true. Let $J = (a, b)$. By Zariski's Unique Factorization and One-to-One Correspondence Theorems, $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$, where $I_1, \cdots, I_l$ are distinct simple m-primary integrally closed ideals in R and $\mu_i \geq 1$ for $i = 1, \cdots, l$, and $\mathrm{Min}(mR[I/b]) = \{q_1, \cdots, q_l\}$ and, upon reordering, v_i is the discrete valuation of $R[I/b]_{q_i}$ for $i = 1, \cdots, l$. $mR[a/b]$ is a prime ideal in $R[a/b]$ of $ht\big(mR[a/b]\big) = 1$ since a, b are a regular sequence on R. Set $T = R[a/b]_{mR[a/b]}$. Then we have

$$\begin{aligned} e(I) &= \sum_{i=1}^{l} \mu_i v_i(I) \qquad \text{by Lemma 3.5} \\ &= e(J) \qquad (J \text{ is a reduction of } I) \\ &= e(JT) \qquad \text{by Lemma 3.6.} \end{aligned}$$

Let T' be the integral closure of T. Then $T' = R[I/b]_{R[a/b]-mR[a/b]}$ is an one-dimensional Noetherian normal domain since $R[a/b] \subseteq R[I/b]$ is integral and $R[I/b]$ is normal. By Corollary of the Krull-Akizuki Theorem ([6], Corollary of Theorem 11.7), there are just a finite number of maximal ideals of T' lying over mT.

Claim. $mR[a/b] = q_i \cap R[a/b]$ for $i = 1, \cdots, l$.

Let $Q_i = q_i \cap R[a/b]$ for $i = 1, \cdots, l$. Then $Q_i \supseteq mR[a/b]$. It is enough to show that $ht(Q_i) = 1$ for $i = 1, \cdots, l$. Suppose that $ht(Q_i) = 2$. Then $R[a/b]/Q_i \cong k[X]/Q_i'$, where Q_i' is the image of Q_i in $k[X]$, since $Q_i \supseteq mR[a/b]$. Hence the 0-dimensional domain $R[a/b]/Q_i$ is a finitely generated k-algebra. By the Nullstellensatz ([1], Corollary 5.24), $R[a/b]/Q_i$ is a finite algebraic extension of k. Since k is an algebraically closed field, $R[a/b]/Q_i = k$. But $(a/b)^* \in R[a/b]/Q_i$, where $(a/b)^*$ is the image of a/b in $R[a/b]/Q_i$, which is a contradition since $(a/b)^*$ is transcendental over k. The proof of the claim is complete.

By the above claim, $q_1T', \cdots, q_lT'$ are all the maximal ideals of T' lying over mT and

$$T' = \bigcap_{i=1}^{l} (T')_{q_iT'} = \bigcap_{i=1}^{l} R[I/b]_{q_i}.$$

By the projection formula ([8], Corollary 1, p. 299),

$$e(JT) = \sum_{i=1}^{l} [T'/q_iT' : T/mT] e(J(T')_{q_iT'}).$$

Since $R[a/b]/mR[a/b] \subseteq R[I/b]/q_i$ have the same quotient fields by the assumption, $T/mT = T'/q_iT'$ for all i. $e(J(T')_{q_iT'}) = v_i(J)$ since $(T')_{q_iT'} = R[I/b]_{q_i}$ is a Rees valuation ring of I with associated valuation v_i for $i = 1, \cdots, l$. $v_i(J) = v_i(I)$ for all i since $J \subseteq I$ is a reduction of I. We have

$$e(JT) = \sum_{i=1}^{l} v_i(I).$$

Hence we have

$$\sum_{i=1}^{l} \mu_i v_i(I) = \sum_{i=1}^{l} v_i(I).$$

That is,

$$\sum_{i=1}^{l} (\mu_i - 1) v_i(I) = 0.$$

Since $v_i(I)$ is positive value and $\mu_i \geq 1$ for all i, $\mu_i = 1$ for all i. The proof is complete. □

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