

INVARIANCE OF THE L_p SPECTRUM FOR HYPOELLIPTIC OPERATORS

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ABSTRACT. We show that the spectra of the L_p -realizations for a class of hypoelliptic (pseudo-)differential operators are independent of p in an interval around $p = 2$ depending on the growth properties of the symbol. For elliptic operators we obtain the classical boundedness interval of Fefferman; in the general case we obtain a smaller interval which is as large as one can possibly expect it to be.

INTRODUCTION

Every differential operator A on Euclidean space naturally induces an unbounded operator A_p on $L_p(\mathbb{R}^n)$. The question to what extent its spectrum depends on p is of considerable interest both in the theory of partial differential equations and mathematical physics.

This problem attracted additional attention when B. Simon conjectured that the spectrum of a Schrödinger operator $-\Delta + V$ is independent of p . Following upon work by Simon and Sigal, Hempel and Voigt [9] showed in 1986 that the conjecture holds for a rather general class of potentials V .

Their result has since been extended in various directions. On the other hand, one also knows many situations where the spectrum *does* depend on p . For several different aspects see e.g. Arendt [2], Davies [4], [5], Davies, Simon, and Taylor [6], Sturm [18].

Hieber [10] recently considered Fourier multipliers on \mathbb{R}^n . Under a hypoellipticity assumption he was able to prove that the L_p -spectrum of each such operator agrees with its L_2 -spectrum for all p in a certain interval around $p = 2$. Through his paper we became convinced that it should be possible to bring in pseudodifferential techniques.

In fact, the problem how the spectrum varies with p also has been considered in connection with algebras of zero-order singular integral and pseudodifferential operators. Here, too, one encounters both phenomena in the literature: There are early results by Widom [20] and Gohberg-Krupnik [8] on p -dependence, while the

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articles Alvarez-Hounie [1], Leopold-Schrohe [14], and Schrohe [16] suggest invariance to a very large extent. Although the question of invertibility is of a quite different type in the bounded context, some of the methods still apply.

In the present article we show that the spectrum of a hypoelliptic (pseudo-) differential operator is independent of p in an interval around $p = 2$. Moreover, this interval is as large as we can realistically expect it to be.

The idea of our proof is rather simple: We show that the L_p -resolvent is a pseudodifferential operator and therefore furnishes an inverse on all spaces where it is continuous. Apart from a mild technical condition, we assume that the operator A is given by a symbol a of order $m > 0$ in Hörmander's class $S_{\rho,\delta}^m$ for $0 \leq \delta < \rho \leq 1$, and that it satisfies a hypoellipticity condition which allows us to construct a parametrix with symbol in $S_{\rho,\delta}^{-m'}$, $m' > 0$. Applying a theorem by Fefferman [7] we see that the parametrix will only extend to a bounded operator on $L_p(\mathbb{R}^n)$, if $|\frac{1}{2} - \frac{1}{p}|n(1 - \rho) \leq m'$. This is precisely the interval where we establish p -invariance. In fact there is a classical counterexample to boundedness if this condition is violated, namely the Fourier multiplier of Hardy-Littlewood-Hirschmann-Wainger; see [7]. It is given by $\text{op}(\exp(i[\xi]^{1-\rho})/[\xi]^{m'})$, where $[\cdot]$ is a smooth positive function on \mathbb{R}^n with $[\xi] = |\xi|$ for large $|\xi|$. Its symbol is an element in $S_{\rho,0}^{-m'}$ yet the operator is unbounded on $L_p(\mathbb{R}^n)$ whenever p falls outside the above range. In the elliptic case, i.e., for $m = m'$, this shows the maximality of the asserted interval.

Concerning L_p -spectra, $1 < p < \infty$, our result both extends and improves that of [10]: For one thing, we cover the case of x -dependent symbols; for another, we get a larger interval of independence if $m' < m$.

A certain draw-back of the pseudodifferential methods is that they require C^∞ -smoothness. It is rather obvious that this assumption can be relaxed; it is much more difficult, however, to say to what extent this is true. This will be the subject of a forthcoming publication. It seems to be a new point, however, that we can guarantee an interval of p -independence in terms of symbol estimates, without requiring self-adjointness, Gaussian estimates, or uniform ellipticity.

1. RESULTS

Let $m \geq m' > 0$, $0 \leq \delta < \rho \leq 1$, and let $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ be a smooth (C^∞) complex-valued function satisfying the following two conditions:

- (i) $|D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$; i.e., a is an element of $S_{\rho,\delta}^m = S_{\rho,\delta}^m(\mathbb{R}^{2n})$.
- (ii) The argument of $a(x, \xi)$ is a bounded continuous function, there is an $R > 0$ such that $a(x, \xi)$ is invertible for $|\xi| \geq R$, and

$$\begin{aligned} |D_\xi^\alpha D_x^\beta a(x, \xi)a^{-1}(x, \xi)| &\leq C_{\alpha\beta} \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}, \\ |a^{-1}(x, \xi)| &\leq C \langle \xi \rangle^{-m'}, \quad |\xi| \geq R. \end{aligned}$$

The symbol a defines the pseudodifferential operator $\text{op } a : \mathcal{S} \rightarrow \mathcal{S}$ by means of the formula

$$(1.1) \quad [\text{op } a]u(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi.$$

As usual, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions, $\hat{u}(\xi) = \int e^{-ix\xi} u(x) dx$ is the Fourier transform of u , and $d\xi = (2\pi)^{-n} d\xi$.

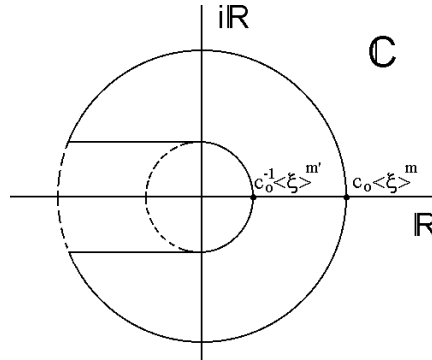


FIGURE 1. The set Ω_0

The symbol a may also take values in the space of $l \times l$ -matrices over \mathbb{C} . In this case, we consider u in (1.1) as an l -vector of functions in \mathcal{S} . For simplicity we shall write $\mathcal{S}, L_p, H_p^s, C_0^\infty$, etc. without referring to either \mathbb{R}^n or l . In the matrix-valued case we replace (ii) by the following condition:

- (ii') There exist a bounded, real-valued continuous function $\theta = \theta(x, \xi)$ and an $R > 0$ such that all eigenvalues of $a(x, \xi)$ are in the domain $\Omega_{x, \xi}$ defined below, and

$$|D_\xi^\alpha D_x^\beta a(x, \xi) \circ (a(x, \xi) - \zeta)^{-1}| \leq C_{\alpha\beta} \langle \xi \rangle^{-\rho|\alpha| + \delta|\beta|}, \quad |\xi| \geq R, \quad \zeta \in \mathbb{C} \setminus \Omega_{x, \xi}.$$

The choice of the domains $\Omega_{x, \xi}$ goes back to Kumano-go [12, Chapter 8]. One lets $\Omega_{x, \xi} = e^{i\theta(x, \xi)}\Omega_0$ where, for $m \geq m' > 0$ and some fixed $c_0 > 1$,

$$\Omega_0 = \left\{ z \in \mathbb{C} : c_0^{-1} \langle \xi \rangle^{m'} < |z| < c_0 \langle \xi \rangle^m, \text{ and } |Im z| > c_0^{-1} \langle \xi \rangle^{m'} \text{ for } Re z < 0 \right\}.$$

In particular it follows from (ii') that $|a(x, \xi)^{-1}| \leq C \langle \xi \rangle^{-m'}$.

For $1 < p < \infty$ we let A_p denote the L_p -realization of $A = \text{op } a$ with $D(A_p) = \{u \in L_p : Au \in L_p\}$. Notice that A_p is a closed operator. The resolvent set of A_p is written $\rho(A_p)$, the spectrum $\sigma(A_p)$.

We shall prove the following theorem.

Theorem 1.1. *Given a symbol a with properties (i) and (ii) or (i) and (ii'), the spectrum of the operators A_p is the same for all $1 < p < \infty$ with $m' \geq n|\frac{1}{2} - \frac{1}{p}|(1 - \rho)$.*

Remark 1.2. (a) For $\rho = 1$, all $1 < p < \infty$ are admissible, since symbols in $S_{1, \delta}^0$ define bounded operators on L_p , cf. Illner [11]. In this case, the additional condition on the existence of a continuous argument function (or its analog in the matrix case) is not necessary; see Lemma 1.3 below.

(b) For $\rho < 1$, the condition $m' \geq n|\frac{1}{2} - \frac{1}{p}|(1 - \rho)$ guarantees the boundedness of operators with symbols in $S_{\rho, \delta}^{-m'}$ on L_p , see Fefferman [7].

Lemma 1.3. *Let $A = \text{op } a$ with an elliptic $a \in S_{1, \delta}^m$, $0 \leq \delta < 1$, $m \geq 0$. Then the spectrum of A_p is independent of p for $1 < p < \infty$.*

Ellipticity means that there is an $R > 0$ such that $a(x, \xi)$ is both invertible whenever $|\xi| \geq R$ and satisfies the estimate $a(x, \xi)^{-1} = O(\langle \xi \rangle^{-m})$.

2. PROOFS

Let us start with the simplest case.

2.1. Proof of Lemma 1.3. The ellipticity of a implies that $D(A_p - \lambda) = H_p^m$, the L_p -based Sobolev space of order m . Invertibility of $A_p - \lambda$ as an unbounded operator on L_p therefore is the same as invertibility of the pseudodifferential operator $(A - \lambda) \circ \text{op} \langle \xi \rangle^{-m}$ as a bounded operator on L_p . It is well-known that the latter is independent of p for $1 < p < \infty$, see e.g. [16, Corollary 1.9].

The proof of Theorem 1.1 is more difficult; it is therefore split up into a series of propositions and lemmata. We write $\Psi_{\rho,\delta}^\mu$ for the space of all operators $B : \mathcal{S} \rightarrow \mathcal{S}$ that can be written as a pseudodifferential operator in the form (1.1) with a symbol $b \in S_{\rho,\delta}^\mu$, $\mu \in \mathbb{R}$. The space $\Psi^{-\infty} = \bigcap_{\mu \in \mathbb{R}} \Psi_{\rho,\delta}^\mu$ is easily seen to be independent of the choice of ρ and δ .

Lemma 2.2. *For every $\lambda \in \mathbb{C}$ there is a parametrix $B_\lambda = \text{op} b_\lambda \in \Psi_{\rho,\delta}^{-m'}$ to $A - \lambda$ with*

$$(2.1) \quad B_\lambda(A - \lambda) = I + R_1, \quad (A - \lambda)B_\lambda = I + R_2$$

and $R_1, R_2 \in \Psi^{-\infty}$. We understand (2.1) as an identity for operators on \mathcal{S} .

Proof. First let a be scalar. For every fixed λ , the symbol $a(x, \xi) - \lambda$ of $A - \lambda$ satisfies conditions (i) and (ii). We obtain the assertion from [12, Chapter 2, Theorem 5.4]. For the matrix-valued case we apply Kumano-go [12, Chapter 8, Theorem 1.2]; see also Remark 2.8 and Theorem 2.9 below. \square

Lemma 2.3. *For $\lambda \in \rho(A_p)$ and $\varphi \in \mathcal{S}$,*

$$(A_p - \lambda)^{-1}\varphi = B_\lambda\varphi - B_\lambda R_2\varphi + R_1(A_p - \lambda)^{-1}R_2\varphi \quad \text{in } \mathcal{S}'.$$

Here, B_λ, R_1 , and R_2 are the pseudodifferential operators from Lemma 2.2.

Proof. This is immediate since every one of these pseudodifferential operators is continuous both on \mathcal{S} and \mathcal{S}' , under the assumption $\delta < 1$. \square

2.4. Outline. According to the composition rules for pseudodifferential operators, $B_\lambda R_2$ is an element of $\Psi^{-\infty}$. We are now going to show that

$$(2.2) \quad R_1(A_p - \lambda)^{-1}R_2 \in \Psi^{-\infty}.$$

Then, in view of Lemma 2.3, we conclude that the restriction $(A_p - \lambda)^{-1}|_{\mathcal{S}}$ coincides with a pseudodifferential operator in $\Psi_{\rho,\delta}^{-m'}$, say C . By Fefferman's theorem [7] the operator C extends to a bounded operator on L_q for all q with $1 < q < \infty$ and $m' \geq n|\frac{1}{2} - \frac{1}{q}|(1 - \rho)$. Denote this extension by C_q . The identity

$$(A_p - \lambda)(A_p - \lambda)^{-1} = I \quad \text{on } L_p$$

shows that $(A - \lambda)C\varphi = \varphi$ for all $\varphi \in \mathcal{S}$ since $C : \mathcal{S} \rightarrow \mathcal{S}$ is continuous. As C_q is closed, C_q maps L_q to $D(A_q)$, and

$$(A_q - \lambda)C_q = I \quad \text{on } L_q.$$

Similarly, $A - \lambda$ maps \mathcal{S} to \mathcal{S} and $C(A - \lambda) = I$ on \mathcal{S} . We conclude that

$$C_q(A_q - \lambda)u = u \quad \text{for all } u \in D(A_q).$$

Hence $C_q = (A_q - \lambda)^{-1}$, so $\lambda \in \rho(A_q)$.

It therefore remains to show (2.2). In order to do this we shall employ the following well-known theorem. For a proof see e.g. Ueberberg [19] or Schrohe [17].

Theorem 2.5. *A scalar linear operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ is a regularizing pseudodifferential operator if and only if for all multi-indices α, β , and all $s, t \in \mathbb{R}$, the iterated commutator $\text{ad}^\alpha x \text{ad}^\beta D_x(T)$ extends to a bounded operator*

$$(2.3) \quad \text{ad}^\alpha x \text{ad}^\beta D_x T : H_p^s \rightarrow H_p^t, \text{ for some } p \text{ with } 1 < p < \infty .$$

If T is an $l \times l$ -matrix of linear operators, then T is regularizing if and only if (2.3) holds for each component. Equivalently we may ask the boundedness of

$$\text{ad}^\alpha x I \text{ad}^\beta D_x I(T) : H_p^s \rightarrow H_p^t.$$

Here I is the $l \times l$ identity matrix. Recall the ad-notation: For a multi-index $\beta = (\beta_1, \dots, \beta_n)$ one lets $\text{ad}^\beta D_x(T) = \text{ad}^{\beta_1} D_{x_1} \dots \text{ad}^{\beta_n} D_{x_n}(T)$ with $\text{ad}^0 D_{x_j}(T) = T$ and $\text{ad}^k D_{x_j}(T) = [D_{x_j}, \text{ad}^{k-1} D_{x_j}(T)]$, $k \geq 1$. Analogously we define $\text{ad}^\alpha x(T)$. The notation extends to expressions of the form $\text{ad}^k S(T)$ for operators S acting on \mathcal{S} or \mathcal{S}' provided that all compositions make sense. Leibniz' rule states that

$$\text{ad}^k S(T_1 T_2) = \sum_{k_1+k_2=k} c_{k_1 k_2} \text{ad}^{k_1} S(T_1) \text{ad}^{k_2} S(T_2).$$

For a pseudodifferential operator $T = \text{opt}$ one has

$$(2.4) \quad \text{ad } x_j(T) = -\text{op}(D_{\xi_j} t) \quad \text{and} \quad \text{ad } D_{x_j}(T) = \text{op } D_{x_j} t.$$

2.6. Reduction. We want to show that, for all multi-indices α, β , and all s, t ,

$$(2.5) \quad \text{ad}^\alpha x \text{ad}^\beta D_x (R_1(A_p - \lambda)^{-1} R_2) : H_p^s \rightarrow H_p^t \text{ is bounded.}$$

The definition shows that $\text{ad}^\beta D_x (R_1(A_p - \lambda)^{-1} R_2)$ is a linear combination of terms of the form $R'_1(A_p - \lambda)^{-1} R'_2$ with $R'_1, R'_2 \in \Psi^{-\infty}$. Hence it is no restriction to assume $\beta = 0$. On the other hand, by Leibniz' rule and (2.4) it is sufficient to consider operators of the form

$$R''_1 \text{ad}^\alpha x (A_p - \lambda)^{-1} R''_2$$

and to show that $\text{ad}^\alpha x (A_p - \lambda)^{-1}$ is bounded in L_p . In order to do this we shall first consider a simple commutator $[x, (A_p - \lambda)^{-1}]$ where x now stands for any of the matrix functions $x_1 I, \dots, x_n I$. In the case $m \leq \rho$ it will be easy to show our result; see Lemma 2.7. For $m > \rho$ we shall use complex powers to reformulate $A_p - \lambda$ as a composition of lower order operators.

Lemma 2.7. *Assume additionally that $m \leq \rho$. Then $[x, A] \in \Psi_{\rho, \delta}^0$, and*

$$\begin{aligned} [x, (A_p - \lambda)^{-1}] &= -(A_p - \lambda)^{-1} [x, (A - \lambda)] (A_p - \lambda)^{-1} \\ &= -(A_p - \lambda)^{-1} [x, A] (A_p - \lambda)^{-1} \\ &= -B_\lambda [x, A] (A_p - \lambda)^{-1} + (A_p - \lambda)^{-1} R_2 [x, A] (A_p - \lambda)^{-1} \end{aligned}$$

in $\mathcal{L}(\mathcal{S}, \mathcal{S}')$. The right-hand side furnishes an extension to an operator in $\mathcal{L}(L_p)$, since $(A_p - \lambda)^{-1} \in \mathcal{L}(L_p)$ and $B_\lambda [x, A], R_2 [x, A] \in \Psi_{\rho, \delta}^{-m'}$ induce operators in $\mathcal{L}(L_p)$.

For $m \leq \rho$, induction therefore shows relation (2.5), and the proof of Theorem 1.1 is complete.

Proof. Choose a function $\varphi \in C_0^\infty$ with $\varphi(x) = 1$ for small $|x|$. For $0 < \varepsilon \leq 1$ let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Then $\{\varphi_\varepsilon : 0 < \varepsilon \leq 1\}$ is a uniformly bounded family in $S_{1,0}^0$. Let $X_\varepsilon = \text{op}(\varphi_\varepsilon(x)x)$. The required identity holds with x replaced by X_ε . Moreover, the symbols of $[X_\varepsilon, A]$ form a bounded subset of $S_{\rho,\delta}^0$. In view of the fact that $X_\varepsilon u \rightarrow xu$ in \mathcal{S} for $u \in \mathcal{S}$ we conclude from Remark 1.2(b) that $[X_\varepsilon, A]v \rightarrow [x, A]v$ in $H_p^{-m'}$ as $\varepsilon \rightarrow 0^+$ for $v \in L_p$. On the other hand,

$$[X_\varepsilon, (A_p - \lambda)^{-1}]u \rightarrow [x, (A_p - \lambda)^{-1}]u$$

in \mathcal{S}' whenever $u \in \mathcal{S}$, so we get the desired identity. □

Remark 2.8. We shall next deal with the case where $m > \rho$ by using complex powers. Conditions (i) and (ii), or (i) and (ii') respectively, are sufficient for the construction in the following theorem.

Moreover, it is easily checked that the conditions are satisfied for $A - \lambda$ whenever they hold for A . We merely have to change the constants R and c_0 in (ii').

Theorem 2.9. *There is a family $\{P_z : z \in \mathbb{C}\}$ of complex powers for $A - \lambda$, satisfying $P_z \in \Psi_{\rho,\delta}^{m, \text{Re}z}$, $\text{Re}z \geq 0$, $P_z \in \Psi_{\rho,\delta}^{m', \text{Re}z}$, $\text{Re}z < 0$, $P_{z_1}P_{z_2} \equiv P_{z_1+z_2}$ modulo $\Psi^{-\infty}$, $z_1, z_2 \in \mathbb{C}$, and $P_k \equiv (A - \lambda)^k$ modulo $\Psi^{-\infty}$, $k \in \mathbb{N}_0$.*

For a proof see Kumano-go [12, Chapter 8].

2.10. Preparation. Let $\{P_z : z \in \mathbb{C}\}$ be as in Theorem 2.9. Choose an integer $N > m/\rho$ and find suitable $R, R', R'' \in \Psi^{-\infty}$ so that $A - \lambda = P_1 + R$ and

$$[x, P_1] = [x, P_{1/N}^N + R'] = \sum_{k=0}^{N-1} P_{k/N} [x, P_{1/N}] P_{(N-1-k)/N} + R''.$$

Proposition 2.11. *With the operators R and $P_{k/N}$ as in 2.10 we have*

$$\begin{aligned} -[x, (A_p - \lambda)^{-1}] &= \sum_{k=0}^{N-1} P_{k/N-1} [x, P_{1/N}] P_{(-1-k)/N} \\ &\quad + \sum_{k=0}^{N-1} (A_p - \lambda)^{-1} S_1 [x, P_{1/N}] P_{(-1-k)/N} \\ &\quad + \sum_{k=0}^{N-1} P_{k/N-1} [x, P_{1/N}] S_2 (A_p - \lambda)^{-1} \\ (2.6) \quad &\quad + (A_p - \lambda)^{-1} (S_1 [x, P_{1/N}] S_2 + [x, R]) (A_p - \lambda)^{-1} \end{aligned}$$

in $\mathcal{L}(\mathcal{S}, \mathcal{S}')$ for suitable operators $S_1, S_2 \in \Psi^{-\infty}$.

Proof. Again choose a function $\varphi \in C_0^\infty$ with $\varphi(x) \equiv 1$ for small $|x|$, let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ and, slightly more refined,

$$X_{\varepsilon, \tilde{\varepsilon}} = \text{op}(x\varphi_\varepsilon(x)\varphi_{\tilde{\varepsilon}}(\xi)), \quad 0 < \varepsilon, \tilde{\varepsilon} \leq 1.$$

Each of the operators $X_{\varepsilon, \tilde{\varepsilon}}$ is regularizing. We have, with suitable $S_1, S_2 \in \Psi^{-\infty}$,

$$\begin{aligned} & -[X_{\varepsilon, \tilde{\varepsilon}}, (A_p - \lambda)^{-1}] \\ = & (A_p - \lambda)^{-1}[X_{\varepsilon, \tilde{\varepsilon}}, P_1 + R](A_p - \lambda)^{-1} \\ = & \sum_{k=0}^{N-1} (A_p - \lambda)^{-1} P_{k/N} [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] P_{(N-1-k)/N} (A_p - \lambda)^{-1} \\ & + (A_p - \lambda)^{-1} [X_{\varepsilon, \tilde{\varepsilon}}, R] (A_p - \lambda)^{-1} \\ = & \sum_{k=0}^{N-1} (A_p - \lambda)^{-1} \{ (A - \lambda) P_{k/N-1} + S_1 \} [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] \{ P_{(-1-k)/N} (A - \lambda) + S_2 \} \\ & \times (A_p - \lambda)^{-1} + (A_p - \lambda)^{-1} [X_{\varepsilon, \tilde{\varepsilon}}, R] (A_p - \lambda)^{-1} \\ = & \sum_{k=0}^{N-1} P_{k/N-1} [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] P_{(-1-k)/N} + \sum_{k=0}^{N-1} (A_p - \lambda)^{-1} S_1 [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] P_{(-1-k)/N} \\ & + \sum_{k=0}^{N-1} P_{k/N-1} [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] S_2 (A_p - \lambda)^{-1} \\ & + (A_p - \lambda)^{-1} (S_1 [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] S_2 + [X_{\varepsilon, \tilde{\varepsilon}}, R]) (A_p - \lambda)^{-1} \end{aligned}$$

for all $0 < \varepsilon, \tilde{\varepsilon} \leq 1$ in $\mathcal{L}(\mathcal{S}, \mathcal{S}')$.

Next fix $\varepsilon > 0$ and write, for the moment, $M_\varepsilon = \text{op}(x\varphi_\varepsilon(x))$. Clearly, $\text{op}(\varphi_{\tilde{\varepsilon}}(\xi))u \rightarrow u$ in \mathcal{S} for $u \in \mathcal{S}$ as $\tilde{\varepsilon} \rightarrow 0^+$. Hence we obtain convergence

$$(2.7) \quad [X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}]u \rightarrow [M_\varepsilon, P_{1/N}]u,$$

$$(2.8) \quad [X_{\varepsilon, \tilde{\varepsilon}}, R]u \rightarrow [M_\varepsilon, R]u$$

in \mathcal{S} for $u \in \mathcal{S}$. The symbols of the commutators $\{[X_{\varepsilon, \tilde{\varepsilon}}, P_{1/N}] : 0 < \tilde{\varepsilon} \leq 1\}$ form a bounded subset of $S_{\rho, \delta}^\mu$ for $\mu = m/N + \delta - \rho$. Applying 1.2(b), convergence in (2.7) will hold in $H_p^{-m' - \mu}$ for $u \in L_p$. Since R is regularizing, a corresponding argument shows that, for (2.8), convergence is in L_p whenever $u \in L_p$. On the other hand $[X_{\varepsilon, \tilde{\varepsilon}}, (A_p - \lambda)^{-1}]u \rightarrow [M_\varepsilon, (A_p - \lambda)^{-1}]u$ in \mathcal{S}' for u in \mathcal{S} , so we get equation (2.6) with x replaced by M_ε . With analogous considerations as before – in particular the fact that the symbols of the commutators $\{[M_\varepsilon, P_{1/N}] : 0 < \varepsilon \leq 1\}$ form a bounded subset of $S_{\rho, \delta}^0$ – we let $\varepsilon \rightarrow 0^+$ and obtain the assertion. \square

Notice that we may *not* apply the above computation with $X_{\varepsilon, \tilde{\varepsilon}}$ replaced by x : It is not clear, for example, that $(A_p - \lambda)^{-1}[x, P_1 + R](A_p - \lambda)^{-1}$ maps \mathcal{S} to \mathcal{S}' , even though $[X_{\varepsilon, \tilde{\varepsilon}}, P_1 + R]u \rightarrow [x, P_1 + R]u$ in \mathcal{S} .

Corollary 2.12. $[x, (A_p - \lambda)^{-1}] : \mathcal{S} \rightarrow \mathcal{S}'$ extends to an operator in $\mathcal{L}(L_p)$.

Proof. Each of the four summands on the right-hand side of (2.6) extends to an operator in $\mathcal{L}(L_p)$. Indeed, the first is an element of $\Psi_{\rho, \delta}^{-m'}$ hence is L_p bounded by Fefferman’s theorem. In the other summands, the L_p -bounded operator $(A_p - \lambda)^{-1}$ is composed with a regularizing pseudodifferential operator. \square

2.13. Conclusion. In 2.6 we had reduced the problem to showing that, for each multi-index α , the operator $\text{ad}^\alpha x (A_p - \lambda)^{-1}$ is bounded in L_p . For $|\alpha| = 1$, the assertion has been shown. On the other hand, the identity in Proposition 2.11

shows that the analysis of higher order commutators reduces to this case. Hence the proof is complete.

Remark 2.14. It is possible to avoid the condition on the argument function in certain other instances. For example suppose that $m - m' \leq \rho$. Then we can show, just as is Lemma 2.7, that

$$\begin{aligned} [(A_p - \lambda)^{-1}, x] &= B_\lambda[x, A]B_\lambda + (A_p - \lambda)^{-1}R_1[x, A]R_2(A_p - \lambda)^{-1} \\ &\quad + B_\lambda[x, A]R_2(A_p - \lambda)^{-1} + (A_p - \lambda)^{-1}R_1[x, A]B_\lambda. \end{aligned}$$

Hence the first commutator is bounded on L_p , and so are its iterates, by induction.

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