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# COUNTING ELLIPTIC PLANE CURVES WITH FIXED $j$-INVARIANT

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ABSTRACT. The number of degree $d$ elliptic plane curves with fixed $j$-invariant passing through $3d-1$ general points in $\mathbf{P}^2$ is computed.

## 0. SUMMARY

Let $N_d$ be the number of irreducible, reduced, nodal, degree $d$ *rational plane curves* passing through $3d-1$ general points in the complex projective plane $\mathbf{P}^2$. The numbers $N_d$ satisfy a beautiful recursion relation ([K-M], [R-T]):

$$N_1 = 1,$$

$$\forall d > 1, \quad N_d = \sum_{i+j=d,\ i,j>0} N_i N_j \left( i^2 j^2 \binom{3d-4}{3i-2} - i^3 j \binom{3d-4}{3i-1} \right).$$

Let $E_{d,j}$ be the number of irreducible, reduced, nodal, degree $d$ *elliptic plane curves with fixed $j$-invariant $j$* passing through $3d-1$ general points $\mathbf{P}^2$. $E_{d,j}$ is defined for $d \geq 3$ and $\infty \neq j \in \overline{M}_{1,1}$. In this note, the following relations are established:

$$\forall j \neq 0, 1728, \infty,\ E_{d,j} = \binom{d-1}{2} N_d,$$

$$j = 0,\ E_{d,0} = \frac{1}{3}\binom{d-1}{2} N_d,$$

$$j = 1728,\ E_{d,1728} = \frac{1}{2}\binom{d-1}{2} N_d.$$

If $d \equiv 0 \ mod\ 3$, then $3 \nmid \binom{d-1}{2}$. Since $E_{3l,0}$ is an integer, $N_{3l} \equiv 0 \ mod\ 3$ for $l \geq 1$. In fact, a check of values in [DF-I] shows $N_d \equiv 0 \ mod\ 3$ if and only if $d \equiv 0 \ mod\ 3$ for $3 \leq d \leq 12$. P. Aluffi has calculated $E_{3,j}$ for $j < \infty$ in [Al]. Aluffi's results agree with the above formulas.

Thanks are due to Y. Ruan for discussions about Gromov-Witten invariants and quantum cohomology. The question of determining the numbers $E_{d,j}$ was first considered by the author in a conversation with him. In [K-Q-R], the approach of this paper is studied in the genus 2 case where a degeneration to a reducible union

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of rational curves replaces the nodal degeneration considered here. An enumerative formula for genus 2 plane curves with fixed complex structure is derived. A symplectic determination of $E_{d,j}$ was found independently in [I].

## 1. Kontsevich's space of stable maps

1.1. **The quasi-projective subvarieties $U_C(\Gamma, c, \overline{w})$, $U_{j=\infty}(\Lambda, \overline{w})$.** Fix $d \geq 3$ for the entire paper. Let $C$ be a nonsingular elliptic curve or an irreducible, nodal rational curve of arithmetic genus 1. Consider the coarse moduli space of $3d-1$-pointed stable maps from $C$ to $\mathbf{P}^2$ of degree $d \geq 3$, $\overline{M}_{C,3d-1}(\mathbf{P}^2, d)$. The points of $\overline{M}_{C,3d-1}(\mathbf{P}^2, d)$ correspond to stable maps of degree $d$ from nodal, $3d-1$-pointed degenerations of $C$ to $\mathbf{P}^2$. Two such pointed maps are equivalent if they differ by an isomorphism of the pointed domain curves. For convenience, the notation $\overline{M}_C(d) = \overline{M}_{C,3d-1}(\mathbf{P}^2, d)$ will be used. Let $S_d = \{1, 2, \ldots, 3d-1\}$ be the marking set. Constructions of $\overline{M}_C(d)$ can be found in [A], [B-M], [K], and [F-P]. Let $\Gamma$ be a tree consisting of a distinguished vertex $c$, $k \geq 0$ *other* vertices $v_1, \ldots, v_k$, and $3d-1$ marked legs. Let $0 \leq e \leq d$. Weight the vertex $c$ by $e$. Let $w_1, \ldots, w_k$ be non-negative integral weights of the vertices $v_1, \ldots, v_k$ satisfying:

$$e + w_1 + \cdots + w_k = d.$$

Denote the weighting by $\overline{w} = (e, w_1, \ldots, w_k)$. The marked, weighted tree with distinguished vertex $(\Gamma, c, \overline{w})$ is stable if the following implication holds for all $1 \leq i \leq k$:

$$w_i = 0 \quad \Rightarrow \quad valence(v_i) \geq 3.$$

Two marked, weighted trees with distinguished vertex $(\Gamma, c, \overline{w})$ and $(\Gamma', c', \overline{w}')$ are isomorphic if there is an isomorphism of marked trees $\Gamma \to \Gamma'$ sending $c$ to $c'$ and respecting the weights.

A quasi-projective subvariety $U_C(\Gamma, c, \overline{w})$ of $\overline{M}_C(d)$ is associated to each isomorphism class of stable, marked, weighted tree with distinguished vertex $(\Gamma, c, \overline{w})$. The subvariety $U_C(\Gamma, c, \overline{w})$ consists of stable maps $\mu : (D, p_1, \ldots, p_{3d-1}) \to \mathbf{P}^2$ satisfying the following conditions. The domain $D$ is equal to a union:

$$D = C \cup \mathbf{P}^1_1 \cup \cdots \cup \mathbf{P}^1_k.$$

The marked, weighted dual graph with distinguished vertex of the map $\mu$ is isomorphic to $(\Gamma, c, \overline{w})$. The distinguished vertex of the dual graph of $\mu$ corresponds to the (unique) component of $D$ isomorphic to $C$. Weights of the dual graph of $\mu$ are obtained by the degree of $\mu$ on the components. Note $U_C(\Gamma, c, \overline{w}) = \emptyset$ if and only if $e = 1$.

Let $(\Gamma, c, \overline{w})$ be a stable, marked, weighted tree with distinguished vertex. Assume $e \neq 1$. The dimension of $U_C(\Gamma, c, \overline{w})$ is determined as follows. If $e \geq 2$, then

$$dim\ U_C(\Gamma, c, \overline{w}) = 6d - 2 - k.$$

If $e = 0$, then

$$dim\ U_C(\Gamma, c, \overline{w}) = 6d - k$$

(where $k$ is the number of non-distinguished vertices of $\Gamma$). These calculations are straightforward.

Let $C$ be a nonsingular elliptic curve. Every stable map in $\overline{M}_C(d)$ has domain obtained by attaching a finite number of marked trees to $C$. By the definition of tree and map stability:

$$\bigcup_{(\Gamma,c,\overline{w})} U_C(\Gamma,c,\overline{w}) \;=\; \overline{M}_C(d).$$

Let $C$ be an irreducible, 1-nodal rational curve. The quasi-projective varieties $U_C(\Gamma,c,\overline{w})$ do not cover $\overline{M}_C(d)$. The curve $C$ can degenerate into a simple circuit of $\mathbf{P}^1$'s. Let $\Lambda$ be a graph with 1 circuit ($1^{st}$ Betti number equal to 1, no self edges), $k \geq 1$ vertices $v_0,\dots,v_k$, and $3d-1$ marked legs. Note the different vertex numbering convention. At least 2 vertices are required to make a circuit, so $k \geq 1$. Let $w_0, w_1, \dots, w_k$ be non-negative, integral weights summing to $d$. The marked, weighted graph with 1 circuit $(\Lambda,\overline{w})$ is stable if each zero weighted vertex has valence at least 3. A quasi-projective subvariety $U_C(\Lambda,\overline{w})$ of $\overline{M}_C(d)$ is associated to each isomorphism class of stable, marked, weighted graph with 1 circuit $(\Lambda,\overline{w})$. $U_C(\Lambda,\overline{w})$ consists of stable maps with marked, weighted dual graphs isomorphic to $(\Lambda,\overline{w})$. The union

$$\bigcup_{(\Gamma,c,\overline{w})} U_C(\Gamma,c,\overline{w}) \;\cup\; \bigcup_{(\Lambda,\overline{w})} U_C(\Lambda,\overline{w}) \;=\; \overline{M}_C(d)$$

holds by the definition of stability.

Finally, the dimensions of the loci $U_C(\Lambda,\overline{w})$ will be required. Let $(\Lambda,\overline{w})$ be a stable, marked, weighted graph with 1 circuit. Let $c_1,\dots,c_l$ be the unique circuit of vertices of $\Lambda$. Let $e$ be the sum of the weights of the circuit vertices. $U_C(\Lambda,\overline{w}) = \emptyset$ if and only if $e=1$. If $e \geq 2$, then

$$dim\ U_C(\Lambda,\overline{w}) = 6d-2-k.$$

If $e=0$, then

$$dim\ U_C(\Lambda,\overline{w}) = 6d-k$$

(where $k+1$ is the total number of vertices of $\Lambda$). Again, these results are straightforward.

1.2. **The component $\overline{W}_1(d)$.** Let $\overline{M}_1(d) = \overline{M}_{1,3d-1}(\mathbf{P}^2,d)$ be Kontsevich's space of $3d-1$-pointed stable maps from genus 1 curves to $\mathbf{P}^2$. There is canonical morphism

$$\pi : \overline{M}_1(d) \to \overline{M}_{1,1}$$

obtained by forgetting the map and all the markings except $1 \in S_d$ (the $3d-1$ possible choices of marking in $S_d$ all yield the same morphism $\pi$). Let $j \in \overline{M}_{1,1}$. By the definitions of the moduli spaces, there is a canonical bijection of points:

$$\overline{M}_{C_j}(d) \to \pi^{-1}(j),$$

where $C_j$ is the elliptic curve (possibly nodal rational) with $j$-invariant $j$. Define an open locus $W_1(d) \subset \overline{M}_1(d)$ by

$$[\mu : (D,p_1,\dots,p_{3d-1}) \to \mathbf{P}^2] \in W_1(d)$$

if and only if $D$ is irreducible. Similarly, an open locus $M^{irr}_{1,3d-1} \subset \overline{M}_{1,3d-1}$ of irreducible pointed curves is defined. Since $d \geq 3$, $M^{irr}_{1,3d-1}$ is a nonsingular fine moduli space with a universal curve. There is a canonical forgetful map $\rho : W_1(d) \to M^{irr}_{1,3d-1}$. It is easily seen that $W_1(d)$ is an open set of a tautological $\mathbf{P}^{3d-1}$-bundle over the universal Picard variety of degree $d$ *line bundles* over $M^{irr}_{1,3d-1}$. Therefore, $W_1(d)$ is a nonsingular irreducible variety of dimension $6d-1$ and the morphism $\rho$ is smooth. Let $\overline{W}_1(d)$ be the closure of $W_1(d)$ in $\overline{M}_1(d)$.

Let $\pi_W : W_1(d) \to \overline{M}_{1,1}$ be the restriction of $\pi$. Let $\pi_{M^{irr}} : M^{irr}_{1,3d-1} \to \overline{M}_{1,1}$ be the canonical forgetful map. Then, $\pi_W = \pi_{M^{irr}} \circ \rho$. It is well known that $\pi^{-1}_{M^{irr}}(j)$ is a nonsingular irreducible divisor in $M^{irr}_{1,3d-1}$ for $0, 1728 \neq j \in \overline{M}_{1,1}$. The scheme theoretic inverse images $\pi^{-1}_{M^{irr}}(0)$ and $\pi^{-1}_{M^{irr}}(1728)$ are easily seen to be irreducible divisors of multiplicities 3 and 2 respectively. Since $\rho$ is smooth with irreducible fibers, $\pi^{-1}_W(j)$ is a nonsingular irreducible divisor in $W_1(d)$ for $j \neq 0, 1728$ and a divisor of multiplicity $3, 2$ for $j = 0, 1728$ respectively.

## 2. A deformation result

Let $\Phi$ be a stable, marked, weighted tree with distinguished vertex $c$ determined by the data: $k = 1$, $(e, w_1) = (0, d)$. There are $2^{3d-1}$ isomorphism classes of such $\Phi$ determined by the marking distribution. Let $j \in \overline{M}_{1,1}$. The dimension of $U_j(\Phi)$ is $6d-1$. A point $[\mu] \in U_j(\Phi)$ has domain $C_j \cup \mathbf{P}^1$. There are $3d-1$ dimensions of the map $\mu|_{\mathbf{P}^1} : \mathbf{P}^1 \to \mathbf{P}^2$. The incidence point $p = C_j \cap \mathbf{P}^1$ moves in a 1-dimensional family on $\mathbf{P}^1$. The remaining $3d-1$ markings move in $3d-1$ dimensions on $C_j$ and $\mathbf{P}^1$ (specified by the marking distribution). $6d-1 = 3d-1+1+3d-1$. A technical result is needed in the computation of the numbers $E_{d,j}$.

**Lemma 1.** *Let $I(\Phi, j) = \overline{W}_1(d) \cap U_j(\Phi) \subset \overline{M}_1(d)$. The dimension of $I(\Phi, j)$ is bounded by dim $I(\Phi, j) \leq 6d-3$.*

*Proof.* Let $[\mu] \in I(\Phi, j)$ be a point. Let $D = C_j \cup \mathbf{P}^1$ be the domain of $\mu$ as above. The following condition will be shown to hold: the linear series on $\mathbf{P}^1$ determined by $\mu|_{\mathbf{P}^1}$ has vanishing sequence $\{0, \geq 2, *\}$ at the incidence point $p = C_j \cap \mathbf{P}^1$. The existence of a point with vanishing sequence $\{0, \geq 2, *\}$ is a 1-dimensional condition on the linear series. The condition that the incidence point $p$ has this vanishing sequence is an additional 1-dimensional constraint on $p$. Therefore, the dimension of $I(\Phi, j)$ is at most $6d-1-1-1 = 6d-3$. The vanishing sequence $\{0, \geq 2, *\}$ is equivalent to $d(\mu|_{\mathbf{P}^1}) = 0$ at $p$.

It remains to establish the vanishing sequence $\{0, \geq 2, *\}$ at $p$. This result is easily seen in explicit holomorphic coordinates. Let $\triangle_t$ be a disk at the origin in $\mathbb{C}$ with coordinate $t$. Let $\eta : \mathcal{E} \to \triangle_t$ be a flat family of curves of arithmetic genus 1 satisfying:

(i) $\eta^{-1}(0) \stackrel{\sim}{=} C_j$.
(ii) $\eta^{-1}(t \neq 0)$ is irreducible, reduced, and (at worst) nodal.

For each $1 \leq i \leq d$, let $\mathcal{G}_i = \mathcal{H}_i \subset \mathcal{E}$ be the open subset of $\mathcal{E}$ on which the morphism $\eta$ is *smooth*. Consider the fiber product:

$$X = \mathcal{G}_1 \times_{\triangle_t} \cdots \times_{\triangle_t} \mathcal{G}_d \times_{\triangle_t} \mathcal{H}_1 \times_{\triangle_t} \cdots \times_{\triangle_t} \mathcal{H}_d.$$

$X$ is a nonsingular open set of the $2d$-fold fiber product of $\mathcal{E}$ over $\triangle_t$. Let $Y \subset X$ be the subset of points $y = (g_1, \ldots, g_d, h_1, \ldots, h_d)$ where the two divisors $\sum g_i$ and $\sum h_i$ are linearly equivalent on the curve $\mathcal{E}_{\eta(y)}$. $Y$ is a nonsingular divisor in $X$.

Let $p \in C_j = \eta^{-1}(0)$ be a nonsingular point of $C_j$. Certainly $p \in \mathcal{G}_i, \mathcal{H}_i$ for all $i$. Let $\gamma : \triangle_t \to \mathcal{E}$ be any local holomorphic section of $\eta$ such that $\gamma(0) = p$. Let $V$ be a local holomorphic field of vertical tangent vectors to $\mathcal{E}$ on an open set containing $p$. The section $\gamma$ and the vertical vector field $V$ together determine local holomorphic coordinates $(t, v)$ on $\mathcal{E}$ at $p$. Let $\phi_V : \mathcal{E} \times \mathbb{C} \to \mathcal{E}$ be the holomorphic flow of $V$ defined locally near $(p, 0) \in \mathcal{E} \times \mathbb{C}$. The coordinate map

$$\psi : (t, v) \to \mathcal{E}$$

is determined by $\psi(t, v) = \phi_V(\gamma(t), v)$.

Local coordinates on $X$ near the point $x_p = (p, \dots, p, p, \dots, p) \in X$ are given by

$$(t, v_1, \dots, v_d, w_1, \dots w_d).$$

The coordinate map is determined by:

$$\psi_X(t, v_1, \dots, v_d, w_1, \dots w_d) =$$

$$(\psi(t, v_1), \dots, \psi(t, v_d), \psi(t, w_1), \dots, \psi(t, w_d)) \in X.$$

Note $x_p \in Y$. Let $f(t, v_1, \dots, v_d, w_1, \dots w_d)$ be a local equation of $Y$ at $x_p$. Since $f$ is identically 0 on the line $(t, 0, \dots, 0, 0, \dots, 0)$,

$$\forall k \geq 0, \quad \frac{\partial^k f}{\partial t^k}|_{x_p} = 0. \tag{1}$$

The tangent directions in the plane $t = 0$ correspond to divisors on the fixed curve $C_j$. Here, it is well known (up to a $\mathbb{C}^*$-factor)

$$\frac{\partial f}{\partial v_i}|_{x_p} = +1, \quad \frac{\partial f}{\partial w_i}|_{x_p} = -1. \tag{2}$$

Equations (1) and (2) are the only properties of $f$ that will be used.

Let $\hat{\eta} : \hat{\mathcal{E}} \to \triangle_t$ be the family obtained by blowing-up $\mathcal{E}$ at $p$ and adding $3d-1$-marking. Let $\mu : \hat{\mathcal{E}} \to \mathbf{P}^2$ be a morphism. Let $\hat{\eta}^{-1}(0) = D = C_j \cup \mathbf{P}^1$. Assume the following conditions are satisfied:

- (i) $\mu$, $\hat{\eta}$, and the $3d-1$ markings determine a family of Kontsevich stable pointed maps to $\mathbf{P}^2$.
- (ii) The markings of $D$ are distributed according to $\Phi$.
- (iii) $deg(\mu|_{C_j}) = 0$, $deg(\mu|_{\mathbf{P}^1}) = d$.

Let $L_1$, $L_2$ be general divisors of $\mu^*(\mathcal{O}_{\mathbf{P}^2}(1))$ that each intersect $\mathbf{P}^1$ transversely at $d$ distinct points. For $1 \leq \alpha \leq 2$, $L_\alpha$ breaks into holomorphic sections $s_{\alpha,1} + \ldots + s_{\alpha,d}$ of $\hat{\eta}$ over a holomorphic disk at $0 \in \triangle_t$. These sections $s_{\alpha,i}$ $(1 \leq \alpha \leq 2,\ 1 \leq i \leq d)$ determine a map $\lambda : \triangle_t \to Y$ locally at $0 \in \triangle_t$. Let an affine coordinate on $\mathbf{P}^1$ be given by $\xi$ corresponding to the normal direction

$$\frac{d\gamma}{dt}|_{t=0} + \xi \cdot V(p). \tag{3}$$

Let $s_{1,i}(0) = \nu_i \in \mathbb{C} \subset \mathbf{P}^1$, $s_{2,i}(0) = \omega_i \in \mathbb{C} \subset \mathbf{P}^1$ be given in terms of the affine coordinate $\xi$. The map $\lambda$ has the form

$$\lambda(t) = (t, \nu_1 t, \dots, \nu_d t, \omega_1 t, \dots, \omega_d t)$$

to first order in $t$ (written in the coordinates determined by $\psi_X$). Equations (1), (2), and the condition $f(\lambda(t)) = 0$ imply

$$\sum_{i=1}^{d} \nu_i = \sum_{1}^{d} \omega_i. \tag{4}$$

$L_1 \cap \mathbf{P}^1$ is a degree $d$ polynomial with roots at $\nu_i$. Condition (4) implies that the sums of the roots (in the coordinates (3)) of general elements of the linear series $\mu|_{\mathbf{P}^1}$ are the same. Therefore, a constant $K$ exists with the following property. If

$$\beta_0 + \beta_1 \xi + \ldots + \beta_{d-1}\xi^{d-1} + \beta_d \xi^d$$

is an element of the linear series $\mu|_{\mathbf{P}^1}$, then $\beta_{d-1} + K \cdot \beta_d = 0$. The vanishing sequence at $\xi = \infty$ is therefore $\{0, \geq 2, *\}$. The point $\xi = \infty$ is the intersection $C_j \cap \mathbf{P}^1$.

Suppose $\tilde{\eta} : \tilde{\mathcal{E}} \to \triangle_t$ is obtained from $\mathcal{E}$ by a sequence of $n$ blow-ups over $p$. The fiber $\tilde{\eta}^{-1}(0)$ is assumed to be $C_j$ union a chain of $\mathbf{P}^1$'s of length $n$. Each blow-up occurs in the exceptional divisor of the previous blow-up. Let $\mathbf{P}$ denote the extreme exceptional divisor. Let $\mu : \tilde{\mathcal{E}} \to \mathbf{P}^2$ be of degree $d$ on $\mathbf{P}$ and degree 0 on the other components of the special fiber $\tilde{\eta}^{-1}(0)$. Let there be $3d-1$ markings as before. It must be again concluded that the linear series on $\mathbf{P}$ has vanishing sequence $\{0, \geq 2, *\}$ at the node.

Let $\gamma$ be section of $\eta$ such that the lift of $\gamma$ to $\tilde{\eta}$ meets $P$. Let the coordinates $(t, v)$ on $\mathcal{E}$ be determined by this $\gamma$ (and any $V$). An affine coordinate $\xi$ is obtained on $\mathbf{P}$ in the following manner. Let $\gamma_\xi$ be the section of $\eta$ determined in $(t, v)$ coordinates by

$$\gamma_\xi(t) = (t, \xi t^n).$$

Let $\tilde{\gamma}_\xi$ be the lift of $\gamma_\xi$ to a section of $\tilde{\eta}$. The association

$$\mathbb{C} \ni \xi \mapsto \tilde{\eta}(0) \in \mathbf{P}$$

is an affine coordinate on $\mathbf{P}$. Let $L_1, L_2$ be divisors in the linear series $\mu$ intersecting $\mathbf{P}$ transversely. As before, $L_\alpha$ breaks into holomorphic sections $s_{\alpha,1}$. Let $s_{1,i} = \nu_i \in \mathbb{C} \subset \mathbf{P}$, $s_{2,i} = \omega_i \in \mathbb{C} \subset \mathbf{P}$. As before, a map $\lambda : \triangle_t \to Y$ is obtained from the sections $s_{\alpha,i}$. In the coordinates determined by $\psi_X$,

$$\lambda(t) = (t, \nu_1 t^n + O(t^{n+1}), \ldots, \nu_d t^n + O(t^{n+1}), \\ \omega_1 t^n + O(t^{n+1}), \ldots, \omega_d t^n + O(t^{n+1})).$$

As before $f(\lambda(t)) = 0$. The term of leading order in $t$ of $f(\lambda(t))$ is

$$(\sum_{i=1}^{d} \nu_i - \sum_{i=1}^{d} \omega_i) \cdot t^n.$$

This follows from equations (1) and (2). The vanishing sequence $\{0, \geq 2, *\}$ is obtained as before.

By definition, an element $[\mu] \in I(\Phi, j)$ can be obtained as the special fiber of family of Kontsevich stable maps where the domain is a smoothing of the node $p$. After resolving the singularity in the total space at the node $p$ by blowing-up, a family $\tilde{\mathcal{E}}$ is obtained. The above results show the linear series on $\mathbf{P}^1$ has vanishing sequence $\{0, \geq 2, *\}$ at $p$. □

The markings play no role in the preceding proof. An identical argument establishes the following:

**Lemma 2.** *Let $\Phi$ be a stable, marked, weighted tree with distinguished vertex satisfying $e = 0$ and $w_i = d$ for some $i$. Let $k$ be the number of non-distinguished vertices of $\Phi$. Let $j \in \overline{M}_{1,1}$. Let $I(\Phi, j) = \overline{W}_1(d) \cap U_j(\Phi)$. The dimension of $I(\Phi, j)$ is bounded by $\dim I(\Phi, j) \leq 6d - k - 2$.*

**Lemma 3.** *Let $\Omega$ be a stable, marked, weighted graph with 1 circuit. Let $v_i$ be a non-circuit vertex with weight $w_i = d$ (this implies $e = 0$). Let $k+1$ be the total number of vertices of $\Omega$. Let $I(\Omega,\infty) = \overline{W}_1(d) \cap U_\infty(\Omega)$. The dimension of $I(\Omega,\infty)$ is bounded by dim $I(\Omega,\infty) \leq 6d-k-2$.*

The vanishing sequence $\{0, \geq 2, *\}$ condition reduces the dimensions of $U_C(\Phi)$, $U_\infty(\Omega)$ by 2.

## 3. The numbers $E_{d,j}$

The space of maps $\overline{M}_1(d)$ is equipped with $3d-1$ evaluation maps corresponding to the marked points. For $i \in S_d$, let $e_i : \overline{W}_1(d) \to \mathbf{P}^2$ be the restriction of the $i^{th}$ evaluation map to $\overline{W}_1(d)$. let $\mathcal{L}_i = e_i^*(\mathcal{O}_{\mathbf{P}^2})$ Let

$$Z = c_1(\mathcal{L}_1)^2 \cap \ldots \cap c_1(\mathcal{L}_{3d-1})^2$$

Let $\pi_{\overline{W}} : \overline{W}_1(d) \to \overline{M}_{1,1} \cong \mathbf{P}^1$ be the restriction of $\pi$ to $\overline{W}_1(d)$. Let

$$T = c_1(\pi_{\overline{W}}^*(\mathcal{O}_{\mathbf{P}^1}(1))).$$

Note $\overline{W}_1(d)$ is an irreducible, projective variety of dimension $6d-1$. The top intersection of line bundles on $\overline{W}_1(d)$, $Z \cap T$, is an integer.

**Lemma 4.**

$$\begin{aligned} \forall j \neq 0, 1728, \infty, &\qquad Z \cap T = E_{d,j}\ , \\ j = 0, &\qquad Z \cap T = 3 \cdot E_{d,0}\ , \\ j = 1728, &\qquad Z \cap T = 2 \cdot E_{d,1728}\ . \end{aligned}$$

*Proof.* Via pull-back, lines in $\mathbf{P}^2$ yield representative classes of $c_1(\mathcal{L}_i)$. Therefore $3d-1$ general points in $\mathbf{P}^2$, $\overline{x} = (x_1, \ldots, x_{3d-1})$, determine a representative cycle $Z_{\overline{x}}$ of the the class $Z$. Let $\infty > j \in \overline{M}_{1,1}$. It is first established for a general representative $Z_{\overline{x}}$,

$$Z_{\overline{x}} \cap \pi_{\overline{W}}^{-1}(j) \subset \pi_{\overline{W}}^{-1}(j). \tag{5}$$

Statement (5) is proven by considering the quasi-projective strata of $\overline{M}_{C_j}(d)$.

Note $\pi_W^{-1}(j)$ is the strata $U_{C_j}(\Gamma, c, \overline{w})$ where $(\Gamma, c, \overline{w})$ is the trivial, stable, marked, weighted tree with distinguished vertex. Assume now $(\Gamma, c, \overline{w})$ is not the trivial tree. By the equations for the dimension of $(\Gamma, c, \overline{w})$,

$$dim\ U_{C_j}(\Gamma, c, \overline{w}) \leq 6d-3$$

unless $e = 0$ and $k = 1, 2$. Since the linear series determined by the evaluation maps are base point free, the general intersection (5) will miss all loci of dimension less than $6d-2$.

It remains to consider the trees $(\Gamma, c, \overline{w})$ where $e = 0$ and $k = 1, 2$. If $k = 1$, $(\Gamma, c, \overline{w}) = \Phi$ satisfies the conditions of Lemma (1). By Lemma (1),

$$dim\ I(\Phi, j) \leq 6d-3.$$

Hence, the general intersection (5) will miss all the loci $U_C(\Phi, c, (0, d))$.

If $k = 2$, there are two cases to consider. If there exists a vertex of weight $d$, then $(\Gamma, c, \overline{w}) = \Phi$ satisfies the conditions of Lemma (2). By Lemma (2),

$$dim\ I(\Phi, j) \leq 6d-4.$$

If $w_1+w_2=d$ is a positive partition, then the image of every map $[\mu] \in U_C(\Gamma, c, \overline{w})$ is the union of two rational curves of degrees $w_1$ and $w_2$. No such unions pass through $3d-1$ general points. The proof of claim (5) is complete.

For $\infty > j \neq 0, 1728$, $\pi_W^{-1}(j)$ is a nonsingular irreducible divisor of $W_1(d)$. Since the linear series determined by the evaluation maps are base point free, the general intersection cycle

$$Z_{\overline{x}} \cap \pi_W^{-1}(j) \tag{6}$$

is a reduced collection of $Z \cap T$ points (Bertini's theorem). The general intersection cycle (6) consists exactly of the reduced, nodal, degree $d$ elliptic plane curves with $j$-invariant $j$ passing through the points $\overline{x}$.

The argument for $j = 0, 1728$ is identical except that $\pi_W^{-1}(0)$, and $\pi_W^{-1}(1728)$ are divisors in $W_1(d)$ with multiplicity 3, 2 respectively. These multiplicities arise from the extra automorphisms for $j = 0, 1728$. Therefore the cycle (6) is a collection of

$$\frac{1}{3} \cdot Z \cap T, \ \frac{1}{2} \cdot Z \cap T$$

triple and double points respectively. □

It remains to evaluate $Z \cap T$.

**Lemma 5.** $Z \cap T = \binom{d-1}{2} N_d$.

*Proof.* It is first established for a general representative $Z_{\overline{x}}$,

$$Z_{\overline{x}} \cap \pi_{\overline{W}}^{-1}(\infty) \subset \pi_W^{-1}(\infty). \tag{7}$$

The statement (7) is proven by considering the quasi-projective strata of $\overline{M}_\infty(d)$.

By arguments of Lemma (4), all the loci $U_\infty(\Gamma, c, \overline{w})$ where $(\Gamma, c, \overline{w})$ is not the trivial tree are avoided in the general intersection (7). Only the strata $U_\infty(\Lambda, \overline{w})$ remain to be considered. Let $k+1 \geq 2$ be the total number of vertices of $\Lambda$. By the equations for the dimensions of $U_\infty(\Lambda, \overline{w})$,

$$dim\ U_\infty(\Lambda, \overline{w}) \leq 6d - 2 - k \leq 6d - 3$$

unless all the circuit vertices have weight zero. If all circuit vertices have weight zero, $k+1 \geq 3$. Now

$$dim\ U_\infty(\Lambda, \overline{w}) \leq 6d - k \leq 6d - 3$$

unless $k = 2$.

Only one stable, marked, weighted, graph with 1-circuit $(\Lambda, \overline{w})$ need be considered. Vertices $c_1, c_2$ form a weightless circuit. Vertex $v_3$ is connected to $c_2$ and $w_3 = d$. $(\Lambda, \overline{w}) = \Omega$ satisfies the conditions of Lemma (3). Therefore,

$$dim\ I(\Omega, \infty) \leq 6d - 4.$$

Claim (7) is now proven.

The divisor $\pi_W^{-1}(\infty)$ is nonsingular and irreducible in $W_1(d)$. As above,

$$Z_{\overline{x}} \cap \pi_W^{-1}(\infty) \tag{8}$$

is a reduced collection of $Z \cap T$ points. The general intersection cycle (8) also consists exactly of degree $d$ *maps* of the 1-nodal rational curve $C_\infty$ passing through $\overline{x}$. The image of such a map must be one of the $N_d$ degree $d$, nodal, rational plane curves passing through $\overline{x}$. The number of distinct birational maps (up to isomorphism) from $C_\infty$ to a $\binom{d-1}{2}$-nodal plane curve is exactly $\binom{d-1}{2}$. Therefore, $Z \cap T = \binom{d-1}{2} N_d$. □

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