PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3735–3739
S 0002-9939(97)04190-7

HOMOTOPY FIXED POINTS FOR CYCLIC p-GROUP ACTIONS

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(Communicated by Thomas Goodwillie)

ABSTRACT. The homotopy fixed point p-compact groups for cyclic p-group actions on nonabelian connected p-compact groups are not homotopically discrete.

1. INTRODUCTION

It is a classical result that cyclic groups acting on nonabelian compact connected Lie groups have no isolated fixpoints [2, Lemme 1, p. 46]:

Theorem 1.1. *Let X be a nonabelian connected compact Lie group equipped with an action of a cyclic group G. Then the identity component of the fixed point group X^G is nontrivial.*

In this note we prove an analog for p-compact groups of this statement. First, we need a few concepts.

Suppose that X is a p-compact group [4] with classifying space BX and that G is a finite group.

Definition 1.2. A G-action on X is a sectioned fibration

$$BX \longrightarrow (BX)_{hG} \underset{Ba}{\overset{\longrightarrow}{\longleftarrow}} BG$$

over BG with fibre BX.

If G is a finite p-group, it is known [4, 5.8] that each component of the section space $(BX)^{hG}$ is the classifying space of a p-compact group. We define the *homotopy fixed point p-compact group for the G-action* to be the p-compact group

$$X^{hG} = \Omega((BX)^{hG}, Ba)$$

whose classifying space is the component containing the section Ba.

Having introduced these concepts, we can now formulate the main result of this note. (A connected p-compact group is nontrivial if its classyfying space is noncontractible.)

Theorem 1.3. *Let X be a nonabelian connected p-compact group equipped with an action of a cyclic p-group G. Then the identity component of the homotopy fixed point p-compact group X^{hG} is nontrivial.*

Received by the editors July 10, 1996.
1991 *Mathematics Subject Classification.* Primary 55P35.
Key words and phrases. Fixed point, p-compact group, Lefschetz number, Weyl group.

The following consequence of this theorem, whose proof relies on a Lefschetz number calculation, is immediate.

Corollary 1.4. *Let $\nu\colon G \to Y$ be a monomorphism of a cyclic p-group into a (not necessarily connected) p-compact group Y which is not a p-compact toral group. Then the identity component of the centralizer $C_Y(G)$ of ν is nontrivial.*

The self-centralizing diagonal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of $\mathrm{O}(n)$ shows that noncyclic subgroups may have discrete centralizers.

Corollary 1.4 plays an important role in the proof of the main result of [9].

2. A Lefschetz number calculation

Let X be a connected p-compact group, $G = \mathbb{Z}/p^r, r \geq 0$, a cyclic p-group and

$$BX \longrightarrow (BX)_{hG} \underset{Ba}{\rightleftarrows} BG$$

an action of G on X. The homotopy fixed point p-compact group X^{hG} is the section space of the fibrewise looping

$$X \longrightarrow X_{hG} \rightleftarrows BG$$

of the G-action. Consider the associated monodromy homomorphisms

$$G \to \mathrm{Aut}_*(BX), \tag{1}$$

$$G \to \mathrm{Aut}_*(X) \tag{2}$$

of G into the groups of based homotopy classes of based self-homotopy equivalences of the fibres and the induced representations

$$G \to \mathrm{Aut}\, H^*_{\mathbb{Q}_p}(BX), \tag{3}$$

$$G \to \mathrm{Aut}\, H^*_{\mathbb{Q}_p}(X) \tag{4}$$

of G in the p-adic rational cohomology algebras. Of course, representation (4) induces yet another representation

$$G \to QH^*_{\mathbb{Q}_p}(BX) \tag{5}$$

of G in the graded vector space of indecomposables.

Let $Bg\colon BX \to BX$ and $g\colon X \to X$ be the self-homotopy equivalences induced by a generator $g \in G$. We shall compute the Lefschetz number

$$\Lambda(X;G) = \sum (-1)^i \operatorname{trace} H^i_{\mathbb{Q}_p}(g)$$

for the action of G on X in terms of the irreducible summands of the representation (5).

Recall [4, §4] that the cyclic group G admits $r+1$ essentially distinct irreducible representations $\rho_0, \rho_1, \ldots, \rho_r$ over the p-adic numbers. Here, ρ_0 is the trivial representation and $\rho_i, 1 \leq i \leq r$, is the composition of the reduction map $G = \mathbb{Z}/p^r \to \mathbb{Z}/p^i$ with the action of $\mathbb{Z}/p^i$, regarded as the group of p^ith roots of unity, on the extension field $\mathbb{Q}_p(\omega_i)$ of $\mathbb{Q}_p$ by a primitive p^ith root of unity ω_i. The dimension of $\rho_i, 1 \leq i \leq r$, is $[\mathbb{Q}_p(\omega_i) : \mathbb{Q}_p] = p^i - p^{i-1}$.

Proposition 2.1. *Suppose that the G-representation $QH^*_{\mathbb{Q}_p}(BX)$ contains the irreducible representation ρ_i with multiplicity n_i, $0 \le i \le r$. Then*

$$\Lambda(X;G) = \begin{cases} p^{n_1+\cdots+n_r} & \text{if } n_0 = 0, \\ 0 & \text{if } n_0 \neq 0 \end{cases}$$

is the Lefschetz number for the action of G on X. In particular, $\Lambda(X;G) = 0$ if and only if G fixes a nonzero vector of $QH^*_{\mathbb{Q}_p}(BX)$.

Proof. Note that the monodromy action (2) of G on $X = \Omega BX$ is the looping of the monodromy action (1) on BX and that the Eilenberg–Moore spectral sequence provides a functorial isomorphism between the graded object $\mathrm{Gr}(H^*_{\mathbb{Q}_p}(X))$ associated to a filtration of $H^*_{\mathbb{Q}_p}(X)$ and the exterior algebra $E(\Sigma^{-1}QH^*_{\mathbb{Q}_p}(BX))$ on the desuspension of $QH^*_{\mathbb{Q}_p}(BX)$. Combining this with the isomorphism

$$QH^*_{\mathbb{Q}_p}(BX) \cong n_0\rho_0 \oplus n_1\rho_1 \oplus \cdots \oplus n_r\rho_r$$

of G-representations induces yet another isomorphism

$$\mathrm{Gr}(H^*_{\mathbb{Q}_p}(X)) \cong E(\Sigma^{-1}\rho_0)^{\otimes n_0} \otimes E(\Sigma^{-1}\rho_1)^{\otimes n_1} \otimes \cdots \otimes E(\Sigma^{-1}\rho_r)^{\otimes n_r}$$

of G-representations. By the additivity [4, 4.12] of traces in exact sequences, then, the Lefschetz number

$$\Lambda(X;G) = \prod_{i=0}^{r} \Lambda_i^{n_i}$$

where Λ_i is the trace for the action of G on $E(\Sigma^{-1}\rho_i)$.

Since $E(\Sigma^{-1}\rho_0)$ is the trivial representation, $\Lambda_0 = 0$.

When $i > 0$, we pass to an algebraic closure of $\mathbb{Q}_p$. Then ρ_i splits into 1-dimensional representations and we see that $\Lambda_i = \Phi_i(1)$ where Φ_i is the characteristic polynomial for g acting on ρ_i or, equivalently, for ω_i acting on $\mathbb{Q}_p(\omega_i)$. Hence Φ_i is the p^ith cyclotomic polynomial so $\Phi_i(1) = p$ and the proposition follows. □

The consequence below is evident if we recall [4, 4.5, 5.7, 5.10] that the Lefschetz number $\Lambda(X;G)$ computes the Euler characteristic of X^{hG} and that a p-compact group is homotopically discrete if it looks so in p-adic rational cohomology.

Corollary 2.2. *The following conditions are equivalent:*

(1) *X^{hG} has a nontrivial identity component.*
(2) *$\chi(X^{hG}) > 0$.*
(3) *$\Lambda(X;G) > 0$.*
(4) *G fixes a nonzero vector of $QH^*_{\mathbb{Q}_p}(BX)$.*

The proof of Theorem 1.3 has now been reduced to the following

Lemma 2.3. *Suppose that X is nonabelian (i.e. not a p-compact torus). Then G fixes a nonzero vector of $QH^*_{\mathbb{Q}_p}(BX)$.*

Proof. Let $T \to X$ be a maximal torus with Weyl group W. The dual weight lattice $L = \pi_2(BT)$ is then a $\mathbb{Z}_p[W]$-module whose rationalization $L \otimes \mathbb{Q}$ exhibits W as a reflection group over $\mathbb{Q}_p$. The action of G on the symmetric invariants $\mathrm{Sym}((L \otimes \mathbb{Q})^*)^W \cong H^*_{\mathbb{Q}_p}(BX)$ factors [4, 8.11, 9.5], [8, §3] through $N(W)/W$ where $N(W)$ is the normalizer of $W < \mathrm{Aut}(L \otimes \mathbb{Q})$.

Suppose first that X is almost simple, i.e. [5, 1.6] that the center of X is finite and that $L \otimes \mathbb{Q}$ is a simple $\mathbb{Q}_p[W]$-module. Then the reflection group W is one of the irreducible reflection groups on the Shephard–Todd–Clark–Ewing list as presented e.g. in [6, p. 165]. The list provides information about the indecomposables of the invariant ring in that the degrees of each reflection group are given.

If $p > 2$, the list shows that $\dim_{\mathbb{Q}_p} QH^i_{\mathbb{Q}_p}(BX) < p-1$ for all i. (In fact $QH^i_{\mathbb{Q}_p}(BX)$ has dimension ≤ 2 with dimension 2 occurring only in case 2a (where the degrees given in [6] are incorrect) and in case 19, neither of which are realizable for $p = 3$.) Since a nontrivial p-adic representation of a cyclic p-group requires at least $p-1$ dimensions, G must act trivially on all of $QH^*_{\mathbb{Q}_p}(BX)$ (which is nonzero if X is nontrivial [4, 5.10]).

The case $p = 2$ requires separate treatment. The only irreducible 2-adic reflection groups are the classical Coxeter groups together with group number 24 of rank 3, $W = \mathbb{Z}/2\mathbb{Z} \times \mathrm{GL}_3(\mathbb{F}_2)$, realized by DI(4) [3]. If W is one of the classical Coxeter groups, the effect of an element of the normalizer $N(W)$ on the degree 4 invariants is multiplication by u^2, $2u^2$, or $3u^2$, where $u \in \mathbb{Q}_2^*$ is a 2-adic unit [7, 1.7]. Since -1 doesn't have this form, the 1-dimensional G-representation $H^4_{\mathbb{Q}_p}(BX) = QH^4_{\mathbb{Q}_p}(BX)$ is the trivial one. Generators for the ring of invariant polynomials of the unique nonclassical 2-adic reflection group are [1, p. 101]

$$\begin{aligned} y_8 &= x_1x_2^3 + x_2x_3^3 + x_3x_1^3, \\ y_{12} &= \det\left(\frac{\partial^2 y_8}{\partial x_i \partial x_j}\right), \\ y_{28} &= \det\begin{pmatrix} \frac{\partial^2 y_8}{\partial x_i \partial x_j} & \frac{\partial y_{12}}{\partial x_i} \\ \frac{\partial y_{12}}{\partial x_j} & 0 \end{pmatrix}, \end{aligned}$$

where the subscript on the variable y denotes the dimension of the corresponding indecomposable cohomology class. Note that if an element of $N(W)$ takes y_8 to its opposite, then also y_{12} is taken to its opposite but y_{28} remains fixed. Thus any element of 2-power order in $N(W)/W$ must fix either y_8 or y_{28} (considered as elements of $H^*_{\mathbb{Q}_p}(BX)$).

This proves the lemma for all almost simple p-compact groups.

Next suppose that X is simply connected and nontrivial. Then there exist, by the splitting theorem [5], almost simple p-compact groups $X_1, \ldots, X_n$ with dual weight lattices $L_1, \ldots, L_n$ and Weyl groups $W_1, \ldots, W_n$ such that $X \cong X_1 \times \cdots \times X_n$ and $L \cong L_1 \times \cdots \times L_n$ as $W \cong W_1 \times \cdots \times W_n$-modules. The efffect of Bg on $H^*_{\mathbb{Q}_p}(BX) = \bigotimes H^*_{\mathbb{Q}_p}(BX_i)$ has, cf. [8, 3.5], the form

$$H^*_{\mathbb{Q}_p}(Bg) = (A_1 \otimes \cdots \otimes A_n) \circ \sigma$$

where A_i is an automorphism of $H^*_{\mathbb{Q}_p}(BX_i)$, $1 \leq i \leq n$, and σ is a permutation within the isomorphism classes of these algebras. Hence

$$QH^*_{\mathbb{Q}_p}(Bg) = (QA_1 \oplus \cdots \oplus QA_n) \circ \sigma$$

on $QH^*_{\mathbb{Q}_p}(BX) = \bigoplus QH^*_{\mathbb{Q}_p}(BX_i)$. There are now essentially two distinct cases to consider. Namely, the case where σ is trivial and the case where σ is a cyclic permutation of p-power order > 1. The first case was treated above and in the second case, $QH^*_{\mathbb{Q}_p}(Bg)$ fixes the diagonal. Hence the fixed point vector space $QH^*_{\mathbb{Q}_p}(BX)^G$ is nontrivial for any nontrivial simply connected p-compact group X.

Finally, up to isogeny any connected p-compact group has the form $X \times S$ [11, 5.4] where X is simply connected and S is a p-compact torus and any automorphism is a product of an automorphism of X with an automorphism of S [10, 4.3]. Hence

$$QH^*_{\mathbb{Q}_p}(BX \times BS)^G \cong QH^*_{\mathbb{Q}_p}(BX)^G \oplus QH^*_{\mathbb{Q}_p}(BS)^G$$

is nontrivial if X is nontrivial. □

We conclude this note with the easy proof of Corollary 1.4.

Proof of Corollary 1.4. Let π be the component group and X the identity component of Y. The p-compact group extension

$$X^{hG} \to C_Y(G) \to C_\pi(G)$$

shows that X^{hG} and $C_Y(G)$ have isomorphic identity components. □

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