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WEAKLY CONVERGENT SEQUENCE COEFFICIENT IN KÖTHE SEQUENCE SPACES

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Abstract. In this paper, we have discussed the weakly convergent sequence coefficient in Köthe sequence spaces with (e_n) as their boundedly complete basis. Using those results, we can easily calculate the weakly convergent sequence coefficient in Orlicz sequence spaces.

1. Introduction

Our aim is to calculate the weakly convergent sequence coefficient in the *Köthe* sequence spaces. The weakly convergent sequence coefficient concerned with normal structure is an important geometric parameter (see [1], [2], [10], [8]). It was introduced by Bynum (see [1]). In the sequel, X denotes a *Banach* space and $S(X)$ denotes the unit sphere of X. l^0 stands for the space of all infinite real sequences. Let N and R be the set of natural numbers and the set of real numbers, respectively.

Definition 1. The *weakly convergent sequence coefficient of* X, denoted $\mathrm{WCS}(X)$, is defined as follows:

$$\begin{aligned}\mathrm{WCS}(X) = \sup\{k :\ & \text{for each weakly convergent sequence } \{x_n\}_{n=1}^{\infty}, \\ & \text{there exists some } y \in \mathrm{co}(\{x_n\}) \text{ such that } k \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\};\end{aligned}$$

here $\mathrm{co}(\{x_n\})$ denotes the convex hull of the elements of $\{x_n\}_{n=1}^{\infty}$.

It is easy to see that $1 \le \mathrm{WCS}(X) \le 2$ (see [1]). The notion of normal structure has been introduced by *Brodskiĭ and Millman* in [4] connected with the fixed point theory. For a bounded subset A of X, the *Chebyshev* self-radius of A is the number

$$r(A) = \inf\{\sup\{\|x - y\| : y \in A\} : x \in \mathrm{co}(A)\}.$$

Definition 2. A Banach space X is said to have *normal structure* if $r(A) < \mathrm{Diam}(A)$ for every non-singleton bounded subset A of X.

It is known that a Banach space with normal structure has the fixed point property (see [1], [3], [4], [9]), and every reflexive Banach space with $\mathrm{WCS}(X) > 1$

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has normal structure (see [1]). For a sequence $\{x_n\}_{n=1}^{\infty}$ of X, we define

$$A(\{x_n\}) = \lim_{n\to\infty} \sup\{\|x_i - x_j\| : i, j \geq n, i \neq j\}$$

and

$$A_1(\{x_n\}) = \lim_{n\to\infty} \inf\{\|x_i - x_j\| : i, j \geq n, i \neq j\}.$$

Definition 3. A sequence $\{x_n\}$ is said to be an *asymptotic equidistant sequence* if $A(\{x_n\}) = A_1(\{x_n\})$ (see [2]).

The result that $\mathrm{WCS}(X) = \inf\{A(\{x_n\})\colon \{x_n\}$ a sequence in $S(X)$ and $x_n \xrightarrow{w} 0\} = \inf\{A(\{x_n\})\colon \{x_n\}$ an asymptotic equidistant sequence in $S(X)$ and $x_n \xrightarrow{w} 0\}$ is obtained in [2].

Definition 4. A Banach space X is said to have *weakly uniformly normal structure* if $\mathrm{WCS}(X) > 1$ (see [7]).

A complete normed sequence space X is called a *Köthe* sequence space if $x = (x(1), x(2), \dots) \in l^0$ and $y = (y(1), y(2), \dots) \in X$ such that $|x(i)| \leq |y(i)|$ for all $i \in N$, then $x \in X$ and $\|x\| \leq \|y\|$.

Definition 5. A map $M : R \to R$ is called an *Orlicz* function if it satisfies the following conditions:

(1) M is even, continuous, convex, and $M(0) = 0$ if and only if $u = 0$;

(2) $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$ and $\lim_{u\to 0} \frac{M(u)}{u} = 0$.

For every Orlicz function M we define the *complementary* function $M^* : R \to R$ by the formula $M^*(\nu) = \sup_{u>0}\{u|\nu| - M(u)\}$ for every $\nu \in R$.

An Orlicz sequence space l_M generated by M is defined as follows:

$$l_M = \{x \in l^0 : R_M(kx) = \sum M(kx(i)) < \infty \text{ for some } k > 0\}.$$

The Orlicz sequence spaces l_M are considered as Banach spaces equipped with the *Luxemburg* norm $\|x\|_{(M)} = \inf\{k > 0 : R_M(\frac{x}{k}) \leq 1\}$, or with the *Orlicz-Amemiya* norm $\|x\|_M = \inf\{\frac{1}{k}(1 + R_M(kx)) : k > 0\}$.

To simplify notation, we put $l_{(M)} = (l_M, \|\ \|_{(M)})$ and $l_M = (l_M, \|\ \|_M)$.

Definition 6. We say an Orlicz function M satisfies the δ_2-*condition* ($M \in \delta_2$, for short) if there exist constants $K \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u) \quad \text{for every } |u| \leq u_0.$$

For more details about Orlicz spaces, we refer the reader to [5], [9], [11] or [12].

2. Results

Theorem 1. *Let X be a Köthe sequence space with $\{e_n\}$ as its boundedly complete basis, where $e_n = (0, 0, \dots, 0, \overset{nth}{1}, 0, \dots)$. Then*

$$\mathrm{WCS}(X) = \inf\left\{A(\{x_n\})\colon x_n = \sum_{i=I_{(n-1)}+1}^{I_n} x_n(i)e_i, x_n \xrightarrow{w} 0, I_1 < I_2 < \cdots\right\}.$$

Proof. Let $d = \inf\{A(\{x_n\}) : x_n = \sum_{i=I_{(n-1)}+1}^{I_n} x_n(i)e_i, x_n \xrightarrow{w} 0, I_1 < I_2 < \cdots\}$. We need only show that $\mathrm{WCS}(X) \geq d$.

For any $\varepsilon > 0$, by the definition of $\mathrm{WCS}(X)$, there exists an asymptotic equidistant sequence $\{x_n\}$ in $S(X)$ with $x_n \xrightarrow{w} 0$ such that $A(\{x_n\}) < \mathrm{WCS}(X) + \varepsilon$.

By $x_n \xrightarrow{w} 0$, we have that

$$x_n(i) \to 0 \qquad \text{as } n \to \infty \ (i = 1, 2, \dots). \tag{1}$$

Let $v_1 = x_1$ and take $I_1 \in N$ so that $\|\sum_{i=I_1}^{\infty} v_1(i)e_i\| < \varepsilon$. By (1) we can choose $N_1 \in N$ such that

$$\left\|\sum_{i=1}^{I_1} x_n(i)e_i\right\| < \varepsilon$$

whenever $n \geq N_1$.

Let $v_2 = x_{N_1}$ and take $I_2 > I_1$ such that $\|\sum_{i=I_2+1}^{\infty} v_2(i)e_i\| < \varepsilon$. By (1), we can find $N_2 > N_1$ so that for $n \geq N_2$,

$$\left\|\sum_{i=1}^{I_2} x_n(i)e_i\right\| < \varepsilon.$$

Let $v_3 = x_{N_2}$ and take $I_3 > I_2$, so that $\|\sum_{i=I_3+1}^{\infty} v_3(i)e_i\| < \varepsilon$.

In such a way, we can obtain a sequence $\{I_n\}$ of positive integers with $0 < I_1 < I_2 < \dots$, and a subsequence $\{v_n\}$ of $\{x_n\}$ satisfying $A(\{x_n\}) = A(\{v_n\})$ and

$$\left\|\sum_{i=1}^{I_{n-1}} v_n(i)e_i\right\| < \varepsilon, \quad \left\|\sum_{i=I_n+1}^{\infty} v_n(i)e_i\right\| < \varepsilon, \qquad n = 1, 2, 3, \dots,$$

where $I_0 = 0$.

Hence, for any $n < m$ we have

$$\begin{aligned}
&\|v_n - v_m\| \\
&\quad = \left\|\left(\sum_{i=1}^{I_{(n-1)}} + \sum_{i=I_{(n-1)}+1}^{I_n} + \sum_{i=I_n+1}^{I_{(m-1)}} + \sum_{i=I_{(m-1)}+1}^{I_m} + \sum_{i=I_m+1}^{\infty}\right)(v_n(i) - v_m(i))e_i\right\| \\
&\quad \geq \left\|\left(\sum_{i=I_{(n-1)}+1}^{I_n} + \sum_{i=I_{(m-1)}+1}^{I_m}\right)(v_n(i) - v_m(i))e_i\right\| \\
&\qquad - \left\|\left(\sum_{i=1}^{I_(n-1)} + \sum_{i=I_n+1}^{I_{(m-1)}} + \sum_{i=I_m+1}^{\infty}\right)(v_n(i) - v_m(i))e_i\right\| \\
&\quad \geq \left\|\sum_{i=I_{(n-1)}+1}^{I_n} v_n(i)e_i + \sum_{i=I_{(m-1)}+1}^{I_m} v_m(i)e_i\right\| - 2\varepsilon - 6\varepsilon \\
&\quad \geq \left\|\sum_{i=I_{(n-1)}+1}^{I_n} u_n(i)e_i + \sum_{i=I_{(m-1)}+1}^{I_m} y_m(i)e_i\right\| - 8\varepsilon,
\end{aligned}$$

where $u_n = \sum_{i=I_{n-1}+1}^{I_n} v_n(i)e_i / \|\sum_{i=I_{(n-1)}+1}^{I_n} v_n(i)e_i\|$, so $\|u_n\| = 1$.

Thus $A(\{u_n\}) \le A(\{v_n\}) + 8\varepsilon \le A(\{x_n\}) + 8\varepsilon \le \mathrm{WCS}(X) + 9\varepsilon$. In view of the arbitrariness of $\varepsilon > 0$, we have $d \le \mathrm{WCS}(X)$.

□

Theorem 2. *Orlicz sequence spaces $l_{(M)}$ (or l_M) has the weak uniform normal structure if and only if $M \in \Delta_2$.*

Proof. Necessity. In the case of $X = l_M$, for any $0 < \varepsilon < 1/2$, by $\mathrm{WCS}(X) = 1$ there exists an asymptotic equidistant sequence $\{x_n\}$ in $S(X)$ with $x_n \xrightarrow{w} 0$ such that $1 \le A(x_n) \le 1 + \varepsilon$.

Without loss of generality, we may assume $1 - \varepsilon \le \|x_n - x_1\|_M \le 1 + \varepsilon$.

By $\|x_n\|_M \ge 1/2$ and $M \in \Delta_2$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{\infty} M(x_n(i)) \ge \delta. \tag{2}$$

Hence, when $n > I_1$, we get

$$\begin{aligned}
1 + \varepsilon &\ge \|x_n - x_1\|_M \\
&= \left(1 + \sum_{i=1}^{\infty} M(k_n(x_n(i) - x_1(i)))\right)/k_n \\
&= \left(1 + \sum_{i=1}^{\infty} M(k_n(x_1(i)))\right)/k_n + \sum_{i=1}^{\infty} M(k_n(x_n))/k_n \\
&\ge \left(1 + \sum_{i=1}^{\infty} M(k_n(x_1(i)))/k_n\right) + \delta \\
&\ge \|x_1\|_M + \delta \ge 1 + \delta.
\end{aligned}$$

This contradiction shows $M \notin \Delta_2$.

Sufficiency. By $M \notin \Delta_2$, for any $\varepsilon > 0$ there is a $u > 0$ such that $u < \varepsilon$ and $\varepsilon M((1+\varepsilon)u) > M(u)$. Setting $v = (1+\varepsilon)u$, we get $M(v/(1+\varepsilon)) < \varepsilon M(v)$.

Since $v < 2\varepsilon$, we can find a positive integer m such that

$$1 - M(2\varepsilon) < mM(v) \le 1.$$

Take $c \ge 0$ satisfying $mM(v) + M(c) = 1$. Then $M(c) < M(2\varepsilon)$. Put

$$\begin{aligned}
x_1 &= \big(c, \overbrace{v, \dots, v}^{m}, 0, 0, \dots\big), \\
x_2 &= \big(\overbrace{0, \dots, 0}^{m}, c, \overbrace{v, \dots, v}^{m}, 0, 0, \dots\big), \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and $y_n = x_n/\|x_n\|_M$. Then $y_n \in S(l_M)$ and $\{y_n\}$ is an asymptotic equidistant sequence.

Next, we will show that $y_n \xrightarrow{w} 0$. It is obvious that $y_n(i) \to 0$ $(i = 1, 2, \dots)$. Since

$$\limsup_{k \to 0} R_M(kx_n)/k = \limsup_{k \to 0} R_M(kx_1)/k = 0,$$

we have $x_n \xrightarrow{l_{M^*}} 0$ which means the convergence with respect to regular functionals. It is obvious that $\Phi(x_n) = 0$ for any singular functional. Hence, $y_n \xrightarrow{w} 0$.

Finally, we estimate $A(\{y_n\})$. We have

$$\begin{aligned}\|(y_n - y_k)/(1+\varepsilon)\|_M &= \|(x_n - x_k)/(1+\varepsilon)\|x_1\|_M\|_M \\ &\le \|(x_n - x_k)/(1+\varepsilon)\|_M \\ &\le 1 + R_M((x_n - x_k)/(1+\varepsilon)) \\ &= 1 + 2mM(v/(1+\varepsilon)) + 2M(c/(1+\varepsilon)) \\ &\le 1 + 2m\varepsilon M(v) + 2M(c) \\ &< 1 + 2\varepsilon + 2M(2\varepsilon).\end{aligned}$$

This means that $A(\{y_n\}) < (1+\varepsilon)(1+2\varepsilon+2M(2\varepsilon))$, i.e., $\mathrm{WCS}(l_M) = 1$.

The proof of the case $X = l_{(M)}$ is similar to that of $X = l_M$. □

Recalling that Maluta's coefficient $D(X)$ of a Banach space X is defined by [10] $D(X) = \sup\{\limsup d(x_{n+1}, \{x_i\}_{i=1}^n)/A(\{x_n\})\colon x_n$ a bounded nonconstant sequence in $X\}$ and $\mathrm{WCS}(X) = 1/D(X)$ for reflexive spaces X (cf. [13]) and $D(X) = 1$ for nonreflexive spaces X (cf. [10]), we obtain that there exists a Banach space X such that $\mathrm{WCS}(X) \neq D(X)$.

Theorem 3. *If M satisfies the Δ_2-condition, we have*

$$\mathrm{WCS}(l_{(M)}) = \inf\left\{c > 0\colon \sum_{i=1}^n M(u_1/c) = 1/2, \sum_{i=1}^n M(u_i) = 1,\ n = 1, 2, \dots\right\}.$$

Proof. Let $d = \inf\{c > 0\colon \sum_{i=1}^n M(u_i/c) = 1/2,\ \sum_{i=1}^n M(u_i) = 1,\ n = 1, 2, \dots\}$. For any $(u_1, u_2, \dots, u_n)$ satisfying $\sum_{i=1}^n M(u_i) = 1$, define

$$x_m = \big(\overbrace{0, \dots, 0}^{mn}, u_1, u_2, \dots, u_n, 0, \dots\big) \qquad (m = 1, 2, \dots).$$

Obviously, $x_m \xrightarrow{w} 0$ and $\|(x_n - x_k)/c\|_{(M)} = 1$ if $n \neq k$, i.e., $A(\{x_n\}) = c$.

By $M(u) \in \Delta_2$, $\{e_n\}$ is a boundedly complete base of $l_{(M)}$, whence $\mathrm{WCS}(l_{(M)}) \le d$.

On the other hand, by $M \in \Delta_2$, we have $\|x\|_{(M)} = 1$ if and only if $R_M(x_n) = 1$. For any

$$x_n = \sum_{i=I_{(n-1)}+1}^{I_{(n)}} x_n(i) e_i \in S(X), \quad \text{where } I_1 < I_2 < \cdots,$$

we have $x_n \xrightarrow{w} 0$ and $R_M((x_n - x_k)/d) = R_M(x_n/d) + R_M(x_k/d) \ge 1$ $(n \neq k)$. This means that $A(\{x_n\}) \ge d$, i.e., $\mathrm{WCS}(l_{(M)}) \ge d$.

So, $\mathrm{WCS}(l_{(M)}) = \inf\{c > 0\colon \sum_{i=1}^n M(u_i/c) = 1/2,\ \sum_{i=1}^n M(u_i) = 1,\ n = 1, 2, \dots\}$. □

Corollary 1. *For l_p $(1 < p < \infty)$ we have* $\mathrm{WCS}(l_p) = 2^{1/p}$ *(see* [1] *and* [10]*).*

Theorem 4. *If $M \in \Delta_2$, then* $\mathrm{WCS}(l_M) = \inf\{\inf\{c_{x,k} > 0\colon R_M(kx/c_{x,k}) = (k-1)/2\} : x = \sum_{i=1}^n x(i)e_i \in S(l_M)\}$.

Proof. Let $d = \inf\{\inf(c_{x,k} > 0\colon R_M(kx/c_{x,k}) = (k-1)/2\}\colon x = \sum_{i=1}^n x(i)e_i \in S(l_M)\}$.

For any $\varepsilon > 0$, there is an $x = \sum_{i=1}^n x(i)e_i \in S(l_M)$, such that

$$\inf \left\{ c_{x,k} > 0\colon R_M(kx/c_{x,k}) = (k-1)/2\colon x \in \sum_{i=1}^n x(i)e_i \in S(l_M) \right\} < d + \varepsilon.$$

So, there are $k > 1$ and $c_{x,k} < d + \varepsilon$ such that $R_M(kx/c_{x,k}) = (k-1)/2$. Put

$$x_m = \big(\overbrace{0, \dots, 0}^{mn}, u_1, u_2, \dots, u_n, 0, \dots\big) \qquad (m = 1, 2, \dots).$$

By the same method as in the proof of Theorem 3, we have $x_n \xrightarrow{w} 0$.

For any $k \neq n$, observe that

$$\begin{aligned} \|(x_n - x_k)/(d+\varepsilon)\|_M &\le (1 + R_M(k(x_n - x_k)/(d+\varepsilon)))/k \\ &= (1 + 2R_M(kx/(d+\varepsilon)))/k \\ &\le (1 + 2R_M(kx/c_{x,k}))/k = 1. \end{aligned}$$

This means that $\|x_n - x_k\|_M \le d + \varepsilon$, i.e., $A(\{x_n\}) \le d + \varepsilon$. By the arbitrariness of ε, we have $\mathrm{WCS}(l_M) \le d$.

Next, we will prove that $\mathrm{WCS}(l_M) \ge d$. Let $x_n = \sum_{i=I_{(n-1)}}^{I_n} x(i)e_i \in S(l_M)$ ($I_0 = 0$) be an arbitrary equidistant sequence such that $x_n \xrightarrow{w} 0$. Take $k_{n,m} > 0$ such that

$$\|(x_n - x_m)/d\|_M = (1 + R_M(k_{n,m}(x_n - x_m)/d))/k_{n,m}.$$

We will estimate $\|(x_n - x_m)\|_M$ by considering two cases:

I. $k_{n,m} \le 1$. Then

$$\|(x_n - x_m)/d\|_M > 1/k_{n,m} \ge 1,$$

whence, $\|(x_n - x_m)\|_M > d$.

II. $1 < k_{n,m}$. Without loss of generality, we assume that $n < m$. We have

$$\begin{aligned} \|(x_n - x_m)/d\|_M &= (1 + R_M(k_{n,m}(x_n - x_m)/d)) \\ &= (1 + R_M(k_{n,m}x_n/d) + R_M(k_{n,m}x_m/d))/k_{n,m} \\ &\ge (1 + (k_{n,m}-1)/2 + (k_{n,m}-1)/2)/k_{n,m} = 1. \end{aligned}$$

Consequently, $A(\{x_n\}) \ge d$.

By the arbitrariness of $\{x_n\}$, it follows that $\mathrm{WCS}(l_M) \ge d$, which finishes the proof. □

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