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CONTINUOUS MULTIPLICATIVE MAPPINGS ON $C(X)$

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Abstract. Let X and Y be compact Hausdorff topological spaces, and let $C(X)$ and $C(Y)$ be real Banach algebras of all real-valued continuous functions on X and Y, respectively. The general form of continuous multiplicative mappings $\Phi\colon C(X) \to C(Y)$ is given.

Let X and Y be compact Hausdorff topological spaces, and let $C(X)$ and $C(Y)$ be real Banach algebras of all real-valued continuous functions on X and Y, respectively. We denote by $\mathbb{1}_X, \mathbb{0}_X \in C(X)$ the functions defined by $\mathbb{1}_X(x) = 1$ and $\mathbb{0}_X(x) = 0$ for all $x \in X$. It is well known that every unital ring homomorphism (additive and multiplicative mapping sending $\mathbb{1}_X$ into $\mathbb{1}_Y$) $\Phi\colon C(X) \to C(Y)$ is a composition operator, that is, there exists a continuous mapping $\varphi\colon Y \to X$ such that $\Phi(f) = f \circ \varphi$ for every $f \in C(X)$. The mapping Φ is a ring isomorphism if and only if φ is a homeomorphism. This means in particular, that the ring $C(X)$ determines the space X up to a homeomorphism. Even more is true, namely, the algebra $C(X)$ is completely determined by its multiplicative structure. More precisely, $C(X)$ and $C(Y)$ are isomorphic as multiplicative semigroups if and only if the underlying spaces X and Y are homeomorphic. This follows from Milgram's representation of the general bijective multiplicative mappings from $C(X)$ onto $C(Y)$ [3]. An interested reader can find some related results on multiplicative isomorphisms and multiplicative derivations in [2], [4], [5].

In this note we will study multiplicative mappings from $C(X)$ into $C(Y)$ that are not bijective. It seems hopeless to find a general representation theorem for such mappings without imposing any additional assumptions. We believe that besides the bijectivity the continuity assumption is the most natural one. The special case $\Phi\colon C(X) \to \mathbb{R}$ (this means that Y consists of one point only) has been independently treated by Turowicz [6] and Bourgin [1]. If $\Phi\colon C(X) \to \mathbb{R}$ is a continuous multiplicative functional, then the same holds true for the mapping $|\Phi|$. So, in order to solve the problem of determining the general form of continuous multiplicative functionals on $C(X)$ one has to consider first the general form of positive nontrivial continuous multiplicative functionals. Here, the positivity is defined as $\Phi(f) \geq 0$ for all $f \in C(X)$ and the nontriviality means that Φ is neither identically zero nor identically one. Bourgin [1] and Turowicz [6] proved the following result.

Theorem 1. *Let $\Phi\colon C(X) \to \mathbb{R}$ be a positive nontrivial continuous multiplicative functional. Then there exists a positive regular Borel measure μ on X with at most*

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countable support $S = \{s_1, s_2, s_3, \dots\}$ *satisfying* $\mu(\{s_i\}) > 0,\ i = 1, 2, 3, \dots,$ *such that*

$$\Phi(f) = \begin{cases} 0 & \textit{if } f(s_i) = 0 \textit{ for some positive integer } i, \\ e^{\int \log |f| \, d\mu} & \textit{otherwise.} \end{cases} \tag{1}$$

Now, if $\Phi \colon C(X) \to \mathbb{R}$ is an arbitrary continuous multiplicative functional, then we can write Φ as $\Phi(f) = |\Phi|(f)\sigma(f)$ where σ maps $C(X)$ into $\{-1, 0, 1\}$. Of course, one can define the values of σ at functions f satisfying $\Phi(f) = 0$ arbitrarily, but throughout this note we will always assume that $\sigma(f) = 0$ for such functions. Bourgin [1] observed that the value of $\sigma(f)$ is completely determined by the values of function f on S. Turowicz [6] gave an explicit expression of $\sigma(f)$ depending on values sign(f) on S and two countable families of parameters α_i, β_i, $i = 1, 2, 3, \dots$. In his formula the α's are uniquely determined by the functional Φ while the values of parameters $\beta_1, \beta_2, \beta_3, \dots$ depend not only on Φ but also on the ordering of the countable set S.

It seems that the most natural approach when studying continuous multiplicative mappings $\Phi \colon C(X) \to C(Y)$ is to compose Φ with δ_y, $y \in Y$, and then to apply the results on multiplicative functionals. Here, $\delta_y \colon C(Y) \to \mathbb{R}$ is the evaluation at the point $y \in Y$. It turns out that Turowicz's description of the mapping σ is not suitable for this approach because of being nonunique. To avoid this obstacle we will give a different description of σ and then apply this result to find a general form of continuous multiplicative mappings from $C(X)$ into $C(Y)$.

We start with some definitions. Let S be a closed subset of a compact set X, and let $\mathcal{F}$ be any family of open (with respect to the relative topology on S) and closed subsets of S. Then $\mathcal{F}$ is called a *sign-determining family* on S if

$$A \triangle B \in \mathcal{F} \Leftrightarrow \text{exactly one of sets } A \text{ and } B \text{ belongs to } \mathcal{F} \tag{2}$$

for every pair of open and closed subsets A, $B \subset S$. Here, $A \triangle B$ is defined as $A \triangle B = (A \backslash B) \cup (B \backslash A)$. Note that the empty set does not belong to any sign-determining family. Let f be any real-valued continuous function on X. We denote by $N_S(f)$ the set of all points $x \in S$ satisfying $f(x) < 0$. For a given sign-determining family $\mathcal{F}$ on S we define a mapping $\sigma_{\mathcal{F}} \colon C(X) \to \mathbb{R}$ by

$$\sigma_{\mathcal{F}}(f) = \begin{cases} 0 & \text{there exists } x \in S \text{ such that } f(x) = 0, \\ 1 & f(x) \neq 0 \text{ for all } x \in S \text{ and } N_S(f) \notin \mathcal{F}, \\ -1 & f(x) \neq 0 \text{ for all } x \in S \text{ and } N_S(f) \in \mathcal{F}. \end{cases}$$

From $N_S(fg) = N_S(f) \triangle N_S(g)$ it follows that $\sigma_{\mathcal{F}}$ is multiplicative. Note that in the special case that $\mathcal{F}$ is the empty family we get $\sigma_{\mathcal{F}}(f) = 1$ for all functions f without zeroes in S.

Now we are ready to prove our first result on multiplicative functionals.

Theorem 2. *A mapping* $\Phi : C(X) \to \mathbb{R}$ *is a nontrivial continuous multiplicative functional if and only if there exists a positive regular Borel measure* μ *on* X *with at most countable support* $S = \{s_1, s_2, s_3, \dots\}$ *satisfying* $\mu(\{s_i\}) > 0$, $i = 1, 2, 3, \dots,$ *and a sign-determining family* $\mathcal{F}$ *on* S *such that*

$$\Phi(f) = \begin{cases} 0 & \textit{if } f(s_i) = 0 \textit{ for some positive integer } i, \\ \sigma_{\mathcal{F}}(f) e^{\int \log |f| \, d\mu} & \textit{otherwise.} \end{cases} \tag{3}$$

Proof. Let us first assume that Φ is a nontrivial continuous multiplicative functional. According to Theorem 1 we can find a measure μ with the support S satisfying all the desired properties such that the functional $|\Phi|$ is represented by (1). Let A be any open and closed subset of S. Then $\chi_{S\setminus A} - \chi_A \colon S \to \mathbb{R}$ is a continuous function. So, by Tietze's extension theorem there exists a continuous extension g_A of this function over X. Note that the restriction of g_A^2 to S is identically equal to one. Consequently, $(\Phi(g_A))^2 = 1$, and therefore either $\Phi(g_A) = 1$ or $\Phi(g_A) = -1$. We define $\mathcal{F}$ to be the set of all open and closed subsets $A \subset S$ satisfying $\Phi(g_A) = -1$. We have already mentioned that the value of $\Phi(f)$ is completely determined by the values of functions f on S. Hence, $\mathcal{F}$ is well defined. A straightforward computation shows that $\mathcal{F}$ is a sign-determining family. In order to see that (3) is satisfied for all functions from $C(X)$ we take a function $f \in C(X)$ having no zeroes in S. From $\Phi(|f|^2) = \Phi(f^2) = (\Phi(f))^2$ it follows that $\Phi(|f|) = |\Phi(f)|$. It is also clear that $f\,g_{N_S(f)}$ is a strictly positive function on S. Hence, it is sent by Φ into a positive real number, and consequently,

$$\Phi(f) = \sigma_{\mathcal{F}}(f) e^{\int \log|f|\, d\mu}.$$

To prove the converse we assume that a measure μ with the support S and a sign-determining family $\mathcal{F}$ on S satisfy all the assumptions of our theorem. Obviously, the functional Φ given by (3) is nontrivial and multiplicative. We have to prove that it is also continuous at every $f \in C(X)$. Let us assume first that $\Phi(f) = 0$. It follows that there exists $x \in S$ such that $f(x) = 0$. Let g be any function without zeroes in S. The continuity of Φ at f follows from

$$|\Phi(g)| \le e^{\mu(\{x\}) \log|g(x)|} e^{\int_{X\setminus\{x\}} \log(\|f\| + \|g-f\|)\, d\mu}.$$

The mapping

$$f \mapsto \log\Phi(e^f) = \int f\, d\mu, \qquad f \in C(X),$$

is continuous which further yields the continuity of Φ at every positive function f. The value of $\Phi(f)$ depends only on the values of f on the set S, and consequently, Φ is continuous at every function which is positive on S. It follows that $|\Phi|$ is continuous. It remains to see that for every $f \in C(X)$ without zeroes in S there exists a positive number η such that $\|g - f\| < \eta$ implies $\sigma_{\mathcal{F}}(g) = \sigma_{\mathcal{F}}(f)$. If we take $\eta = \min_{x\in S}\{|f(x)|\}$, then $\|g - f\| < \eta$ yields that g is without zeroes in S and $N_S(g) = N_S(f)$ as required. This completes the proof.

In order to get a complete description of continuous multiplicative mappings $\Phi\colon C(X) \to C(Y)$ we need the following definitions. Let X and Y be compact Hausdorff spaces. Assume that we have associated to each $y \in Y$ a closed subset $S_y \subset X$. The mapping $y \mapsto S_y$ is *continuous at the point* $y_0 \in Y$ if

- for every neighbourhood U of S_{y_0} there exists a neighbourhood V of y_0 such that $S_z \subset U$ for every $z \in V$, and
- for every point $x \in S_{y_0}$ and every neighbourhood W of x there exists a neighbourhood T of y_0 such that $S_z \cap W \neq \varnothing$ for every $z \in T$.

The mapping $y \to S_y$ is *continuous* if it is continuous at every point $y_0 \in Y$.

Assume further that we have associated to each $y \in Y$ also a sign-determining family $\mathcal{F}_y$ on S_y. The mapping $y \mapsto \mathcal{F}_y$ is *continuous at* $y_0 \in Y$ if for every open and closed set $A \subset S_{y_0}$ and every pair of open disjoint subsets $U, V \subset X$ satisfying $A \subset U$ and $S_{y_0} \backslash A \subset V$ there exists a neighbourhood W of y_0 such that

$$A \in \mathcal{F}_{y_0} \Leftrightarrow S_z \cap U \in \mathcal{F}_z$$

for every $z \in W$. Of course, this definition makes sense only under the assumption that the mapping $y \mapsto S_y$ is continuous at y_0. Namely, in this case we can find a neighbourhood T of y_0 such that $S_z \subset U \cup V$ for every $z \in T$. It follows that $S_z \cap U$ is an open and closed subset of S_z. As before, we say that the mapping $y \mapsto \mathcal{F}_y$ is *continuous* if it is continuous at every point $y_0 \in Y$.

Finally, let us assume that $\{\mu_y : y \in Y\}$ is a family of positive Borel measures on X. We will call the mapping $y \mapsto \mu_y$ *lower semicontinuous at* $y_0 \in Y$ if for every open $U \subset X$ and every positive real number ε there exists a neighbourhood V of y_0 such that $\mu_z(U) > \mu_{y_0}(U) - \varepsilon$ for every $z \in V$. This mapping will be called *upper semicontinuous at* $y_0 \in Y$ if for every compact $K \subset X$ and every positive real number ε there exists a neighbourhood V of y_0 such that $\mu_z(K) < \mu_{y_0}(K) + \varepsilon$ for every $z \in V$. The mapping $y \mapsto \mu_y$ is *continuous* if it is upper and lower semicontinuous at every point $y_0 \in Y$.

Theorem 3. *Let X and Y be compact Hausdorff topological spaces. A mapping $\Phi\colon C(X) \to C(Y)$ is a continuous multiplicative mapping if and only if*

$$\Phi(f)(y) = \begin{cases} 0 & y \in Y_0, \\ 1 & y \in Y_1, \\ 0 & y \in Z \text{ and } f \text{ has a zero in } S_y, \\ \sigma_{\mathcal{F}_y}(f) e^{\int \log|f|\, d\mu_y} & y \in Z \text{ and } f \text{ has no zeroes in } S_y, \end{cases} \tag{4}$$

where Y_0, Y_1, and Z are disjoint closed subsets of Y such that $Y = Y_0 \cup Y_1 \cup Z$, μ_y is a positive measure on X having all the properties as in Theorem 1 for every $y \in Z$, $\mathcal{F}_y$ is a sign-determining family on the support S_y of μ_y for every $y \in Z$, and the mappings $y \mapsto \mu_y$, $y \mapsto S_y$, $y \to \mathcal{F}_y$ are continuous on Z.

Proof. Let us first assume that Φ is a continuous multiplicative mapping. Clearly, the functions $\Phi(\mathbb{1}_X)$, $\Phi(\mathbb{0}_X) \in C(Y)$ are idempotents. It follows that $Y_0 = \Phi(\mathbb{1}_X)^{-1}(\{0\})$ and $Y_1 = \Phi(\mathbb{0}_X)^{-1}(\{1\})$ are open and closed subsets of Y. Applying the multiplicativity of Φ it is easy to see that the functional $\delta_y \circ \Phi$ is identically equal to zero for every $y \in Y_0$ and similarly, $\delta_y \circ \Phi$ must be identically equal to one for every $y \in Y_1$. In particular, Y_0 and Y_1 are disjoint subsets of Y. Set $Z = Y \backslash (Y_0 \cup Y_1)$. Obviously, the multiplicative functional $\delta_y \circ \Phi$ is nontrivial for every $y \in Z$. According to Theorem 2 we can associate to each $y \in Z$ a positive regular Borel measure μ_y on X with a countable support S_y and a sign-determining family $\mathcal{F}_y$ on S_y such that $\delta_y \circ \Phi$ has a representation of the form (3). We have to show now that the mappings $y \mapsto \mu_y$, $y \mapsto S_y$, and $y \mapsto \mathcal{F}_y$ are continuous.

Let us choose $y \in Z$ and a neighbourhood U of S_y. By Urysohn's lemma there exists a continuous function $f \in C(X)$ with $\operatorname{supp} f \subset U$ such that its restriction to S_y is identically equal to one. It is easy to see that $\Phi(f)(y) = 1$. As $\Phi(f)$ is continuous at y we can find a neighbourhood $V \subset Z$ of y such that $\Phi(f)(z) > 1/2$ for every $z \in V$. This yields that f has no zeroes in S_z for every $z \in V$ which further implies that $S_z \subset U$ for every $z \in V$. Let us now assume that $x \in S_y$, and that W is a neighbourhood of x. Introducing a function $g \in C(X)$ with $\operatorname{supp} g \subset \{x\}^C$ satisfying $g(u) = 1$ for all $u \in W^C$ one can prove similarly as above that $S_z \cap W \neq \varnothing$

for every $z \in \{w \in Z \colon \Phi(g)(w) < 1/2\}$. Hence, the mapping $y \mapsto S_y$ is continuous on Z.

In the next step we will prove that $y \mapsto \mu_y$ is a continuous mapping on Z. Let K be any compact subset of X, ε a positive real number, and let y be any point in Z. The regularity of μ_y implies the existence of an open subset U in X containing K such that $\mu_y(U) < \mu_y(K) + \varepsilon/2$. Using Urysohn's lemma once again we can find a function $g \in C(X)$ with values in $[0,1]$ whose restriction to K is identically equal to one while it is identically equal to zero outside U. Define $f \in C(X)$ by $f = e^g$. Clearly, $\mu_y(K) \le \log \Phi(f)(y) \le \mu_y(U)$. As $\log \Phi(f)$ is continuous at y there exists a neighbourhood $V \subset Z$ such that $\log \Phi(f)(z) - \log \Phi(f)(y) < \varepsilon/2$ for every $z \in V$. It follows that

$$\mu_z(K) = \log e^{\int_K \, d\mu_z} \le \log \Phi(f)(z) < \mu_y(U) + \varepsilon/2 < \mu_y(K) + \varepsilon, \qquad z \in V.$$

Thus, the mapping $y \mapsto \mu_y$ is upper semicontinuous. In almost the same way one can prove it is also lower semicontinuous.

To prove the continuity of $y \mapsto \mathcal{F}_y$ we choose $y \in Z$, an open and closed subset $A \subset S_y$ and disjoint open subsets $U, V \subset X$ such that $A \subset U$ and $S_y \backslash A \subset V$. We will also assume that $A \in \mathcal{F}_y$ since in the case that $A \notin \mathcal{F}_y$ the proof goes through similarly. We can find continuous functions $g,\ h \colon X \mapsto [0,1]$ such that $g_{|A} \equiv 1$, $g_{|U^C} \equiv 0$, $h_{|S_y \backslash A} \equiv 1$, and $h_{|V^C} \equiv 0$. Set $f = h - g$. Clearly, $\Phi(f)(y) = -1$. We already know that the mapping $t \mapsto S_t$, $t \in Z$, is continuous, and therefore, there exists a neighbourhood $W \subset Z$ of y such that $S_z \subset U \cup V$ and $\Phi(f)(z) < -1/2$ for every $z \in W$. It follows that $\sigma_{\mathcal{F}_z}(f) = -1$, or equivalently, $S_z \cap U \in \mathcal{F}_z$ for every $z \in W$. Hence, $y \mapsto \mathcal{F}_y$ is continuous.

To prove the converse statement assume that a mapping Φ is given by (4) and that the sets Y_0, Y_1, Z, and the mappings $y \mapsto \mu_y$, $y \mapsto S_y$, $y \mapsto \mathcal{F}_y$ fulfill all the requirements of Theorem 3. In order to see that Φ is well defined we have to show that the restriction of $\Phi(f)$ to Z is a continuous function for every $f \in C(X)$. So, choose $\mathbb{0}_X \ne f \in C(X)$ and $y \in Z$. By Z_f we will denote the set of all zeroes of f. Set $M = \max_{x \in X} \log |f(x)|$.

The first possibility we will consider is that $\Phi(f)(y) = 0$, or equivalently, $p = \mu_y(Z_f \cap S_y) > 0$. For every positive number $\eta < 1$ we can find a neighbourhood U of $Z_f \cap S_y$ such that $|f(y)| < \eta$ for every $x \in U$. Now, $\mu_y(U) \ge p$. Therefore, the continuity of the mapping $t \mapsto \mu_t$ yields the existence of a neighbourhood $V \subset Z$ of y such that $\mu_z(X) < 2\mu_y(X)$ and $\mu_z(U) > p/2$ for every $z \in V$. Hence, for $z \in V$ such that f has no zeroes in S_z we have

$$\begin{aligned} |\Phi(f)(z)| &= e^{\int_U \log |f(x)| \, d\mu_z} e^{\int_{X \backslash U} \log |f(x)| \, d\mu_z} \\ &\le e^{(\log \eta) \mu_z(U)} e^{M \int_X \, d\mu_z} \le \eta^{p/2} e^{2M \mu_y(X)}. \end{aligned}$$

This shows that $\Phi(f)$ is continuous at y.

The next possibility is that f is positive on S_y. Let ε be any positive number. There exists a neighbourhood U_0 of S_y such that $1/2 \min_{x \in S_y} f(x) < f(u) < 2 \max_{x \in S_y} f(x)$ for every $u \in U_0$. The continuity of the mapping $t \mapsto S_t$ implies the existence of a neighbourhood $W \subset Z$ of y such that $S_z \subset U_0$ for every $z \in W$. Set $K = \max\{2\mu_y(X),\ 1 + \max_{x \in S_y} |\log f(x)|\}$ and choose a positive number $\delta < 7K\mu_y(X)$ such that $|s - \log \Phi(f)(y)| < \delta$ yields $|e^s - \Phi(f)(y)| < \varepsilon$. As

$S_y = \{x_1, x_2, \dots\}$ is a countable set we can find a positive integer N such that

$$\sum_{k>N} \mu_y(\{x_k\}) < \frac{\delta}{7K}.$$

For every $j = 1, 2, 3, \dots, N$ we can find a neighbourhood $U_j \subset U_0$ of x_j such that $|\log f(x) - \log f(x_j)| < \delta/(7K)$ for every $x \in U_j$. Moreover, we can assume that these neighbourhoods are disjoint. As X is normal and μ_y is regular we can also assume that $\mu_y(\bar{U}_j) < \mu_y(\{x_j\}) + 2^{-j}\delta/(14K)$. Because of the upper and lower semicontinuity of the mapping $t \mapsto \mu_t$ we can find for every $j = 1, 2, 3, \dots, N$ a neighbourhood V_j of y such that

$$\mu_y(U_j) - 2^{-j}\frac{\delta}{7K} < \mu_z(U_j) \le \mu_z(\bar{U}_j) < \mu_y(\bar{U}_j) + 2^{-j}\frac{\delta}{14K} < \mu_y(U_j) + 2^{-j}\frac{\delta}{7K}$$

holds true for every $z \in V_j$. Finally, let $V_0 \subset Z$ be a neighbourhood of y such that $|\mu_z(X) - \mu_y(X)| < \delta/(7K)$ for every $z \in V_0$. Denote $W \cap V_0 \cap V_1 \cap \dots \cap V_N$ by V and $U = \bigcup_{j=1}^N U_j$. For $z \in V$ we have $\mu_z(X) < \mu_y(X) + \delta/(7K) < 2\mu_y(X) \le K$ and

$$\begin{aligned}\mu_z(X\backslash U) &< \mu_y(X) + \delta/(7K) - \sum_{j\le N} \mu_z(U_j) \\ &< \mu_y(X) - \mu_y(U) + 2\delta/(7K) < 3\delta/(7K).\end{aligned}$$

Hence, for every $z \in V$ we get

$$\begin{aligned}|\log \Phi(f)(z) - \log \Phi(f)(y)| &= \left| \int_X \log f(x)\, d\mu_z - \int_X \log f(x)\, d\mu_y \right| \\ \le \left| \int_{X\backslash U} \log f(x)\, d\mu_z \right| &+ \left| \int_U \log f(x)\, d\mu_z - \sum_{j\le N} \log f(x_j) \int_{U_j} d\mu_z \right| \\ &+ \sum_{j\le N} |\log f(x_j)|\, |\mu_z(U_j) - \mu_y(U_j)| \\ &+ \left| \sum_{j\le N} \log f(x_j) \int_{U_j} d\mu_y - \int_U \log f(x)\, d\mu_y \right| + \left| \int_{X\backslash U} \log f(x)\, d\mu_y \right| \\ < \max_{x\in S_z\backslash U} |\log f(x)| \mu_z(X\backslash U) &+ \sum_{j\le N} \mu_z(U_j)\delta/(7K) + K\delta/(7K) \sum_{j\le N} 2^{-j} \\ &+ \mu_y(U)\delta/(7K) + \max_{x\in S_y\backslash U} |\log f(x)| \mu_y(S_y\backslash U) < \delta.\end{aligned}$$

Let us now turn to the more general case that f is without zeroes on S_y. As $|\Phi(f)| = \Phi(|f|)$ the previous step shows that $|\Phi(f)|$ is a continuous function at y. To prove that $\Phi(f)$ is also continuous at y we will find a neighbourhood V of y such that the mapping $z \mapsto \sigma_{\mathcal{F}_z}(f)$ is constant on V. The set $N_{S_y}(f)$ is open and closed in S_y. Hence, disjoint open subsets W_1 and W_2 can be found such that $N_{S_y}(f) \subset W_1 \subset \{x \in X : f(x) < 0\}$ and $S_y \backslash N_{S_y}(f) \subset W_2 \subset \{x \in X : f(x) > 0\}$. Let $V_1 \subset Z$ be a neighbourhood of y such that $S_z \subset W_1 \cup W_2$ for every $z \in V_1$. The continuity of the mapping $t \mapsto \mathcal{F}_t$ guarantees the existence of a neighbourhood $V \subset V_1$ of y such that $\sigma_{\mathcal{F}_z}(f) = \sigma_{\mathcal{F}_y}(f)$ for every $z \in V$. This completes the proof of the continuity of $\Phi(f)$ for every $f \in C(X)$.

Once we know that Φ is well defined it is easy to see that it is multiplicative. It remains to prove that Φ is continuous. The compactness of the set Z implies that it is enough to show that for every $f \in C(X)$ and every $\varepsilon > 0$ and every $y \in Z$ we can find a neighbourhood $V \subset Z$ of y and $\delta > 0$ such that $|\Phi(f)(z) - \Phi(g)(z)| < \varepsilon$ for every $z \in V$ and any $g \in C(X)$ satisfying $\|f - g\| < \delta$. So, let us assume that $f \in C(X)$, $y \in Z$, and $\varepsilon > 0$.

The first case we will consider is that $\Phi(f)(y) \neq 0$. Using the continuity of $|f|$ and $t \mapsto S_t$ once again we can find a positive real number a and a neighbourhood $V \subset Z$ of y such that $\inf\{|f(x)| : x \in S_z \text{ and } z \in V\} \geq 2a$. It follows that for $g \in C(X)$ satisfying $\|g - f\| < a$ we have $N_{S_z}(g) = N_{S_z}(f)$ for every $z \in V$. Hence,

$$|\Phi(g)(z) - \Phi(f)(z)| = |e^{\int_X \log|g(x)|\,d\mu_z} - e^{\int_X \log|f(x)|\,d\mu_z}| \tag{5}$$

for such functions $g \in C(X)$. It is easy to verify that $\mu_t(X) \leq \|\log \Phi(e\mathbb{1}_X)\|$ for every $t \in Z$. Applying this fact in (5) one can now easily conclude that for all functions $g \in C(X)$ that are sufficiently close to f the inequality $|\Phi(g)(z) - \Phi(f)(z)| < \varepsilon$ holds true for every $z \in V$.

To finish the proof it remains to consider the case that $\Phi(f)(y) = 0$. The continuity of $\Phi(f)$ at y implies the existence of a neighbourhood $V_1 \subset Z$ of y such that $|\Phi(f)(z)| < \varepsilon/2$ for every $z \in V_1$. We also know that in this case $\mu_y(Z_f \cap S_y) = p > 0$. Take any $0 < \eta < 1/2$ and let U be a neighbourhood of Z_f such that $|f(x)| < \eta$ for every $x \in U$. The mapping $t \mapsto \mu_t$ is continuous. Hence, one can find a neighbourhood $V \subset V_1$ of y such that $\mu_z(U) > p/2$ for every $z \in V$. Let $g \in C(X)$ satisfy $\|g - f\| < \eta$. This yields $|g(x)| \leq \|g - f\| + |f(x)| < 2\eta$ for every $x \in U$. Hence, for such $g \in C(X)$ and any $z \in V$ we have either $\Phi(g)(z) = 0$ or

$$\begin{aligned} |\Phi(g)(z)| &= e^{\int_U \log|g(x)|\,d\mu_z} e^{\int_{X\setminus U} \log|g(x)|\,d\mu_z} \\ &< e^{\log(2\eta)\mu_z(U)} e^{\int_X \log(\|g-f\|+|f(x)|)\,d\mu_z} < (2\eta)^{p/2} e^{\log(\eta+\|f\|)\|\log\Phi(e\mathbb{1}_X)\|}. \end{aligned}$$

Thus, for η small enough it follows that $|\Phi(g)(z) - \Phi(f)(z)| < \varepsilon$ for every $z \in V$.

References

1. D. G. Bourgin, *Multiplicative transformations*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 564–570. MR **12:**421b
2. H. Goldmann and P. Šemrl, *Multiplicative derivations on $C(X)$*, Monatsh. Math. **121** (1996), 189–198. MR **96m:**46041
3. A. N. Milgram, *Multiplicative semigroups of continuous functions*, Duke Math. J. **16** (1949), 377–383. MR **10:**612a
4. P. Šemrl, *The functional equation of multiplicative derivation is superstable on standard Banach algebras*, Integr. Equat. Oper. Th. **18** (1994), 118–122. MR **94k:**47055
5. P. Šemrl, *Isomorphisms of standard operator algebras*, Proc. Amer. Math. Soc. **123** (1995), 1851–1855. MR **95g:**47066
6. A. Turowicz, *Sur les fonctionelles continues et multiplicatives*, Ann. Soc. Polon. Math. **20** (1947), 135–156.

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