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DEHN FILLING, REDUCIBLE 3-MANIFOLDS, AND KLEIN BOTTLES

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ABSTRACT. Let M be a compact, connected, orientable, irreducible 3-manifold whose boundary is a torus. We announce that if two Dehn fillings create reducible manifold and manifold containing Klein bottle, then the maximal distance is three.

1. INTRODUCTION

Let M be a compact, connected, orientable, irreducible 3-manifold such that ∂M is a torus. The *slope* of an essential simple loop on ∂M is its isotopy class, and if π and γ are two slopes on ∂M then $\triangle = \triangle(\pi, \gamma)$ will denote their minimal geometric intersection number. Let $M(\pi)$ denote the manifold obtained from M by π-Dehn filling, that is, by attaching a solid torus V_π to M along ∂M so that the boundary of a meridian disk is identified with π, and similarly for γ.

There are many results on $\triangle(\pi, \gamma)$ for two distinct slopes π and γ on ∂M, for example [BZ1], [CGLS], [Go1], [GLu1], and [Wu1]. Especially Gordon and Luecke [GLu2] have shown that if both $M(\pi)$ and $M(\gamma)$ are reducible, then $\triangle \leq 1$, and Wu [Wu2] and Oh [Oh] proved independently that if M is hyperbolic and $M(\pi)$ is reducible while $M(\gamma)$ contains an incompressible torus, then $\triangle \leq 3$. In this paper we consider the situation where $M(\gamma)$ contains an embedded Klein bottle.

Theorem 1.1. *Let M be a hyperbolic 3-manifold. If $M(\pi)$ is reducible and $M(\gamma)$ contains a Klein bottle, then $\triangle(\pi, \gamma) \leq 3$.*

It is still unknown whether or not the bound 3 is best possible. This result gives us a partial improvement to Theorem 0.1(2) of [BZ2], dealing with reducible and finite Dehn fillings.

Corollary 1.2. *If M is hyperbolic, $M(\pi)$ is reducible and $M(\gamma)$ is a Seifert fiber space over the 2-sphere with three exceptional fibers of orders $2, 2, n$, then $\triangle(\pi, \gamma) \leq 3$.*

We will be following closely the argument in [Oh] and will hereafter assume familiarity with this paper. To obtain a contradiction, we shall suppose that $\triangle(\pi, \gamma) \geq 4$. Let $\widehat{Q}$ be a reducing sphere in $M(\pi)$ which intersects V_π in a family of meridian

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disks. We choose $\widehat{Q}$ so that $Q = \widehat{Q} \cap M$ has the minimal number, say q, of boundary components. Similarly we choose $\widehat{S}$, a Klein bottle in $M(\gamma)$ in such a manner.

By an isotopy of Q, we may assume that Q and S intersect transversely, and $Q \cap S$ has the minimal number of components. Then no circle component of $Q \cap S$ bounds a disk in Q or S, and no arc of $Q \cap S$ is boundary parallel in Q or S. The boundary of a regular neighborhood of $\widehat{S}$ is a torus $\widehat{T}$, meeting V_γ in $t = 2|\widehat{S} \cap V_\gamma|$ points in $M(\gamma)$. Note that $\widehat{T}$ is not necessarily incompressible. Now we obtain a graph G_Q in $\widehat{Q}$ by taking $\widehat{Q} \cap V_\pi$ as its fat vertices and the arcs in $Q \cap T$ as its edges. Similarly we obtain the graph G_T in $\widehat{T}$. Note that each fat vertex of G_Q (G_T) intersects each fat vertex of G_T (respectively G_Q) exactly $\triangle$ times. Number the components of ∂Q $1, 2, \ldots, q$ successively along ∂M, and similarly number the components of ∂T $1, 2, \ldots, t$. In this way each end of each edge of G_Q (G_T) has a label, namely the number of the corresponding fat vertex of G_T (respectively G_Q). When traveling around a fat vertex of G_Q, the labels appear as $1, \ldots, t$ repeated $\triangle$ times and similarly for a fat vertex of G_T. Assigning orientations to $\widehat{Q}$ and $\widehat{T}$ allows us to refer to $+$ and $-$ vertices of G_Q (G_T), according to the sign of the corresponding intersection with the core of V_π (respectively V_γ). If two vertices have the same sign they are called *parallel,* otherwise *antiparallel.* The orientability of Q, T and M give us the following *parity rule* : an edge connects parallel vertices on one graph if and only if it connects antiparallel vertices on the other. As is done above, define the labelled graphs G_Q^S in $\widehat{Q}$ and G_S in $\widehat{S}$ coming from the intersection of Q and S.

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2. Preliminaries

In this section we define the concept of a Scharlemann cycle and a $(k)\, x$-web, and introduce some combinatorial techniques developed in other articles.

Let G be the graph G_Q or G_T, and x a label of G. An *x-edge* in G is an edge with label x at one endpoint. An *x-cycle* is a cycle of x-edges of G such that all the vertices are parallel and all the edges can be oriented so that the tail of each edge has label x. A *Scharlemann cycle* is an x-cycle that bounds a disk face of G. A Scharlemann cycle with exactly two edges is called an *S-cycle.* Note that by construction, G has no face with only one edge.

A $(k)\, x$*-web* is a connected subgraph Σ of G_Q such that all the edges of Σ are x-edges, all the vertices of Σ are parallel, and all but possibly k edges with label x at the vertices of Σ connect them to the vertices of Σ. Such exceptional edges are called ghost edges, and their endpoints in Σ are called ghost vertices. A *great* $(k)\, x$*-web* satisfies the additional condition that there is a component U of $\widehat{Q} - \Sigma$ such that all the vertices of G in $\widehat{Q} - U$ have the same sign.

We hereafter assume that $q \geq 3$ because of Lemma 2.3 of [BZ3]. We have the following lemma from the argument of Proposition 1.3 of [GLi].

Lemma 2.1. *G_T (hence G_S) cannot have q mutually parallel edges.* □

If we assume that $t \geq 4$, then we have the following two lemmas; Lemma 2.2 is Lemmas 2.6 and 2.7 of [Oh] and Lemma 2.3 can be obtained from Lemmas 2.1 - 2.4 of [Wu1].

Lemma 2.2. *(1) G_Q has at most two S-cycles on disjoint label pairs.*

(2) G_Q has at most $t/2+2$ mutually parallel edges connecting parallel vertices. Furthermore, if $t \equiv 2 \ (mod\, 4)$, then G_Q cannot have $t/2+2$ mutually parallel edges connecting parallel vertices. □

Lemma 2.3. *(1) G_T cannot contain two S-cycles on distinct label pairs.*

(2) G_T has at most $q/2+1$ mutually parallel edges connecting parallel vertices. Furthermore, if there are such $q/2+1$ edges, then first two or last two of these edges form an S-cycle in G_T. □

3. Proof of Theorem 1.1 (the case $t \geq 6$)

In this section we prove Theorem 1.1 when $t \geq 6$. As the argument in the case $t = 2$ or 4 is quite different, we handle it separately in Section 4.

We assume familiarity with the terminology of [GLu1, Section 2.1] and the more generalized terminology discussed in [GLu2]. Proposition 3.1 of [Oh], which is the analog of Proposition 3.1 of [GLu2], is still true in our case, indeed it works for any $\triangle \geq 2$. Hence we get the following proposition;

Proposition 3.1. *Either G_Q contains a great $(k)\,x$-web, or for all $\{1, \ldots, q\}$-types, τ, there are at least kt faces of $G_T(L)$ representing τ.*

Let k be the smallest number greater than $\triangle/2$. Now Theorem 1.1 is broken into two cases, which will be carried out in the rest of this section. For a graph Γ, the *reduced graph* $\overline{\Gamma}$ of Γ is defined to be the graph obtained from Γ by amalgamating each family of mutually parallel edges of Γ to a single edge. Note that every family of mutually parallel edges of G_Q has an even number of edges since $\widehat{T}$ is the boundary of a regular neighborhood of $\widehat{S}$.

Theorem 3.2. *G_Q cannot contain a great $(k)\,x$-web.*

Proof. Assume for contradiction that there is a great $(k)\,x$-web Σ in G_Q. We may assume that Σ has no separating edge, for if e is an edge of Σ such that $\Sigma - e$ has two components then one of them is also a great $(k)\,x$-web. Let U be a component of $\widehat{Q} - \Sigma$ such that all the vertices of G_Q in $\widehat{Q} - U$ have the same sign. Let Γ_Q be the subgraph $G_Q - U$ of G_Q. Suppose that U is an n-gon, i.e. $\overline{\Gamma}_Q$ has n boundary vertices. Let v, e and f be the number of vertices, edges and faces of $\overline{\Gamma}_Q$ in $\widehat{Q}$. Since each face of $\overline{\Gamma}_Q$ is a disk with at least 3 sides, we have $2e \geq 3(f-1)+n$. Thus

$$2 = \chi(\widehat{Q}) = v - e + f \leq v - \frac{e}{3} + 1 - \frac{n}{3}.$$

Therefore $2e \leq 6v - 2n - 6$. We distinguish two cases

(1) Some interior vertex y of $\overline{\Gamma}_Q$ has valency at most $\triangle+1$.

There are $\triangle t$ edges in G_Q which are incident to y and connect y to parallel vertices. By Lemma 2.2(2), $\triangle t \leq (\triangle+1)(\frac{t}{2}+2)$, i.e. $t < 7$ and $t \neq 6$, a contradiction.

(2) All interior vertices of $\overline{\Gamma}_Q$ have valency at least $\triangle+2$.

Suppose some boundary vertex y of $\overline{\Gamma}_Q$ which is not a ghost vertex has valency at most $\triangle - 1$. Then at least $(\triangle-1)t+2$ edges (must be even because T is a double covering of S) incident to y in G_Q connect y to parallel vertices. Again $(\triangle-1)t+2 \leq (\triangle-1)(\frac{t}{2}+2)$ by Lemma 2.2(2). Hence $t < 4$. Therefore all boundary vertices of $\overline{\Gamma}_Q$ which are not ghost vertices have valency at least $\triangle$. Similarly each boundary vertex of $\overline{\Gamma}_Q$ which is a ghost vertex with i ghost edges has valency at least $\triangle - i$. Since $\overline{\Gamma}_Q$ has at most k ghost edges, we get the following inequality (here $v-n$ is the number of interior vertices):

$$(1) \qquad (\triangle + 2)(v - n) + \triangle n - k \leq 2e.$$

From two previous inequalities, we finally have $(\triangle-4)v + 6 \leq k$, which contradicts that $k \leq \triangle/2 + 1$.

It was pointed out by the referee that the great (k) x-web Σ might have some separating vertices. Then after cutting along separating vertices, we can choose a subgraph of Σ which is also a great $(k/2+\triangle)$ x-web containing no separating vertex, because there are at least two subgraphs of Σ containing only one separating vertex of Σ. Then we are able to use the following inequality instead of (1):

$$(\triangle + 2)(v - n) + \triangle n - (\frac{k}{2} + \triangle) \leq 2e.$$

Thus $(\triangle-4)v + 6 - \triangle \leq k/2$, which again contradicts that $k \leq \triangle/2 + 1$. □

Lemma 3.3. *G_T contains a Scharlemann cycle.*

Proof. We separate two cases.

(1) Suppose that there is a vertex x of G_Q such that for all labels y at least 2 edges incident to x at y connect x to antiparallel vertices.

Then the conclusion follows from case (1) of the proof of Lemma 5.1 of [Oh].

(2) Here we assume the negation of (1). That is, for each vertex x of G_Q there is a label $y(x)$ such that at least $\triangle - 1$ edges incident to x at $y(x)$ connect x to parallel vertices.

Let Λ_Q be an innermost connected component of the subgraph which is obtained from G_Q by deleting all edges connecting antiparallel vertices and all separating families of mutually parallel edges. Without loss of generality, we assume that in Λ_Q, every vertex, except possibly one, called a ghost vertex, which is an endpoint of a separating family of mutually parallel edges, has valency at least $(\triangle-2)t+2$. Then Λ_Q here is very similar to the graph Γ_Q described in the proof of Theorem 3.2. The same argument as the proof of Theorem 3.2 shows that there is no interior vertex of Λ_Q of valency at most $\triangle+1$ in $\overline{\Lambda}_Q$, and so there is a boundary vertex, y, of Λ_Q which is not a ghost vertex and has valency at most $\triangle-1$ in $\overline{\Lambda}_Q$ (here $\overline{\Lambda}_Q$ has at most one ghost vertex). By using Lemma 2.2(2) twice, $(\triangle-2)t+2 \leq (\triangle-1)(\frac{t}{2}+2)$, i.e. $t \leq 4(\triangle-2)/(\triangle-3)$ and $t \neq 6$. Thus $t = 8$ when $\triangle = 4$. That is, y has valency at least 18 in Λ_Q and at most 3 in $\overline{\Lambda}_Q$. By Lemma 2.2(2) again, these 18 edges consist of 3 families each of which has exactly 6 mutually parallel edges and so contains 2 S-cycles on disjoint label pairs. And adjacent 2 families contain 4 S-cycles on disjoint label pairs, which contradicts Lemma 2.2(1). □

Now we are ready to prove Theorem 1.1 when $t \geq 6$.

Proof of Theorem 1.1. Using Proposition 3.1 and Theorem 3.2, we are able to conclude that for all $\{1, \dots, q\}$-types, τ, there are at least kt faces, i.e. more than $(\triangle/2)t$ faces, of G_T representing τ. Recall that $\widehat{Q}$ is essential in $M(\pi)$ and G_T contains a Scharlemann cycle. Now we can apply the same arguments in the context of Sections 4, 5 and 6 in [GLu2] where they use the facts that $\triangle = 2$ and for every face type τ there are more than p (the number of boundary components of the other planar surface P) faces representing τ. In our case we have more than $(\triangle/2)t$ faces representing τ. In [GLu2] it is shown that there are a vertex v of G_Q

and certain face types such that each face of G_T representing such a type contains an edge incident to v, and this gives rise to too many edges in G_Q incident to v. The theorem follows. □

4. Exceptional cases $t = 2$ or 4

This section will be devoted to proving Theorem 1.1 in the special cases $t = 2$ and $t = 4$. We will make frequent use of the combinatorics of Lemma 2.1 and Lemma 2.3 throughout this section to complete the proof of the main theorem.

Lemma 4.1. *If $t = 2$ then $\triangle \leq 3$.*

Proof. Since G_S is a graph with one vertex, say 1, on a Klein bottle, the number of families of mutually parallel edges is at most 3, i.e. this vertex has valency at most 6 in $\overline{G}_S$. Hence $\triangle < 6$ by Lemma 2.1. Thus there are two cases.

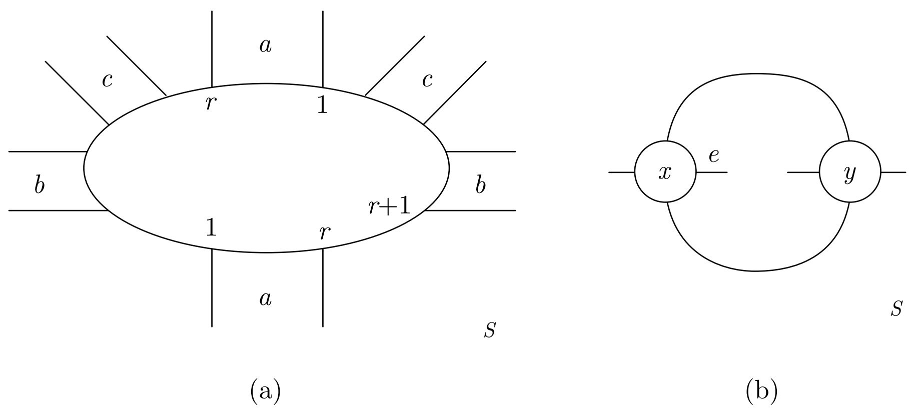

Figure 1

(1) When $\triangle = 4$; G_S looks like Figure 1(a), where $1 < r < q$.

Here only the edges of family a correspond to an orientation-preserving curve on $\widehat{S}$ and hence these edges connect antiparallel vertices of G_Q^S. Each pair of edges labelled x and $r+1-x$ (where $x = 1, 2, \ldots, r$) of family a forms a cycle meeting vertices x and $r+1-x$ in G_Q^S. Choose an innermost such cycle with vertices, say x and $y = r+1-x$, which bounds a disk D such that there is no edge connecting antiparallel vertices in the interior of D, as in Figure 1(b). Since in G_S these edges are incident to 1 at non-adjacent occurrences of the label x, in G_Q^S they are non-adjacent at vertex x. Let e be the edge incident to x that lies in D. Let Γ be the subgraph of G_Q^S which consists of all vertices parallel to x in the interior of D and their connecting edges except e. Then every vertex of Γ, except the one which e is incident to, has even valency, namely 4. This contradicts a property of a graph.

(2) When $\triangle = 5$; We see G_S as shown in Figure 2(a), where $1 < r < \frac{q}{2}$.

Again only the edges of families a_1, a_2 and a_3 correspond to the edges connecting antiparallel vertices of G_Q^S. As in case (1), each pair of edges of families a_1 and a_3 is a cycle meeting two related vertices in G_Q^S. Then there is an innermost cycle with vertices, say x and y, which bounds a disk D such that D does not contain the vertices $1, \ldots, r, \frac{q}{2}+1, \ldots, \frac{q}{2}+r$. Now consider the edges incident to x and y in D. Since both boundary edges of D are incident to 1 at labels x (and y) and of the

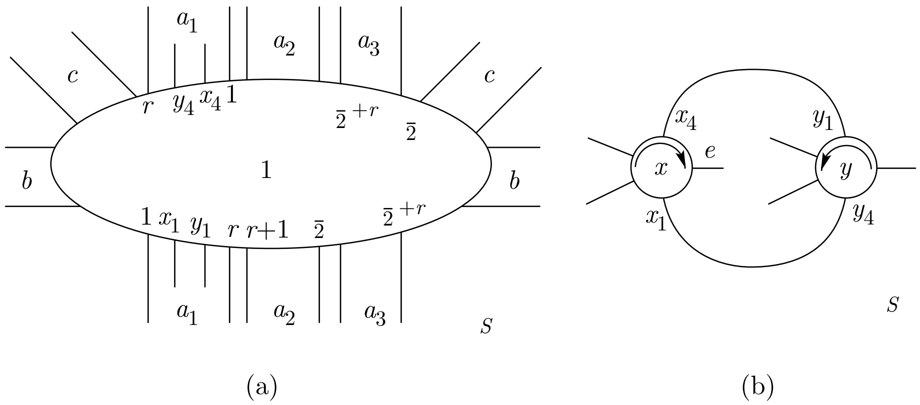

FIGURE 2

same family in G_S, and x and y are antiparallel, if n edges are incident to x in the interior of D then $3-n$ edges are incident to y in the interior of D where $n = 0, 1, 2$ or 3 (as shown in Figure 2(b) when $n = 1$). Thus either x or y has odd valency in the interior of D. Furthermore for each vertex v in the interior of D, only one edge incident to v connects v to an antiparallel vertex (this edge is in family a_2 in G_S); i.e. exactly the other four edges connect v to parallel vertices. As is done in case (1), we can define a subgraph of G_Q^S, only one of whose vertices has odd valency, a contradiction. □

Lemma 4.2. *If* $t = 4$ *then* $\triangle \leq 3$.

Proof. By an Euler characteristic count, $\overline{G}_S$ has at most 6 edges with 2 vertices as shown in Figure 3(a). Hence its double cover $\overline{G}_T$ is the graph illustrated in Figure 3(b) (or the same graph but with the vertices on the middle line labelled $+, -, -, +$) because the first and the third vertices come from the same vertex of $\overline{G}_S$, i.e. they are antiparallel.

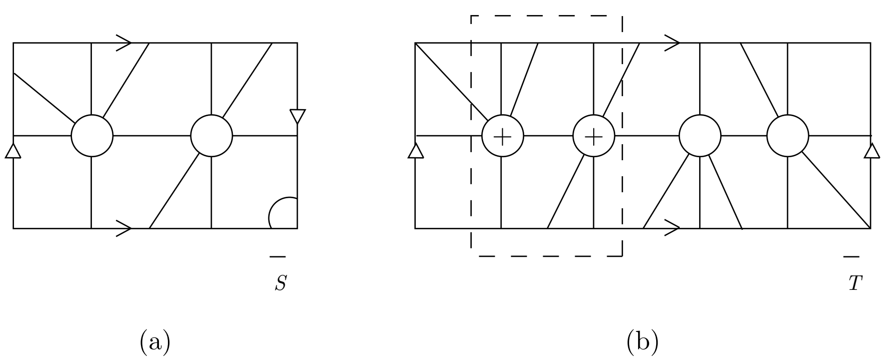

FIGURE 3

We may assume that G_T contains a part of the graph as shown in Figure 4. Lemmas 2.1 and 2.3(2) imply the following inequality:

$$2(q-1) + 4(\frac{q}{2}+1) = 4q + 2 \geq \triangle q.$$

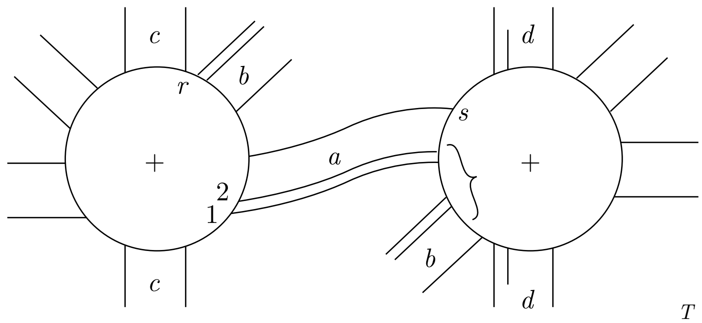

FIGURE 4

Hence $\triangle \leq 4$. Assume that $\triangle = 4$. In Figure 4 r must be odd by the parity rule.

Suppose that families a and b have a total of $q+2$ edges, i.e. each family has $q/2+1$ edges. Lemma 2.3(2) implies that each family contains an S-cycle on the side. Thus family a contains $\{1, 2\}$ or $\{\frac{q}{2}, \frac{q}{2}+1\}$ S-cycle and family b contains $\{\frac{q}{2}+2, \frac{q}{2}+3\}$ or $\{1, 2\}$ S-cycle. If both families contain $\{1, 2\}$ S-cycles, then we have labels 1,2,1 and 2 successively on mark A indicated in Figure 4. It is impossible. Thus G_T contains two S-cycles on distinct label pairs in each family, contradicting Lemma 2.3(1).

Therefore families a and b have q edges, i.e. $r = 1$. Then each family c and d has $q/2+1$ edges. By Lemma 2.3(1) and (2) again, both families contain $\{\frac{q}{2}, \frac{q}{2}+1\}$ S-cycles on the side. If 2 edges on the left side of family d form $\{\frac{q}{2}, \frac{q}{2}+1\}$ S-cycle, then families a and b must have $q-2$ edges. Thus 2 edges on the other side form $\{\frac{q}{2}, \frac{q}{2}+1\}$ S-cycle, i.e. $s = 1$. This implies that the middle 2 edges of family a form an S-cycle which do not have labels $\{\frac{q}{2}, \frac{q}{2}+1\}$, a contradiction. □

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