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ORDER-TOPOLOGICAL SEPARABLE COMPLETE MODULAR ORTHOLATTICES ADMIT ORDER CONTINUOUS FAITHFUL VALUATIONS

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Abstract. We prove that on every separable complete atomic modular ortholattice (i.e. order topological) there exists an order continuous faithful valuation. We also give a construction of the existing order continuous faithful valuation. For separable atomic modular ortholattices we give a necessary and sufficient condition to admit an order continuous faithful valuation and we show that it is equivalent with the condition to have a modular MacNeille completion. We improve one statement on complete metric lattices from Birkhoff's Lattice Theory.

1. Introduction

It is known that the MacNeille completion of an orthomodular lattice (even of a modular ortholattice) need not be orthomodular [1], [6]. The finite or cofinite dimensional subspaces of an infinite dimensional Hilbert space form a modular ortholattice which cannot be embedded into a complete modular ortholattice. On the other hand there are also some positive results about MacNeille completions of orthomodular lattices and modular ortholattices (e.g. [4], [7], [8], [14], [15], [17]).

Another open question is whether on every separable topological complete Boolean algebra there exists an order continuous faithful valuation. On the other hand on every complete irreducible modular ortholattice such order continuous faithful valuation exists ([9], pp. 197–210). The existence of a faithful valuation on complete orthomodular lattice L implies that L is separable and modular (see [19], p. 36). The properties of modular lattices have been carefully investigated by numerous mathematicians, including J. von Neumann who introduced the important study of continuous geometry. J. von Neumann [21] showed that the continuous geometry (i.e. a complete, irreducible, complemented, modular lattice which is join and meet continuous) has a dimension function (real valued and hence it is a valuation, among other properties). The study continued later with removing the irreducibility requirement from the definition of a continuous geometry and proving that a continuous geometry has a dimension function in a more general sense (i.e. not necessarily real-valued, see Maeda's book [13]). It has been obtained that any

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continuous geometry is isomorphic to a sub-direct product of irreducible continuous geometries.

We show that for separable atomic modular ortholattices both those questions, i.e. on the existence of an order continuous faithful valuation and on the modularity of the MacNeille completion are equivalent. Simultaneously we give a necessary and sufficient condition for separable atomic modular ortholattices to admit an order-continuous faithful valuation.

2. Basic notions and facts

Whenever possible we have followed notions and known results in Kalmbach's book [9]. However, let us recall some basic definitions and facts.

Definition 2.1. An *ortholattice* is a lattice L with a least element 0 and a greatest element 1 and with a unary operation $'$ called *orthocomplementation* such that for all $a, b \in L$

(i) $(a')' = a$,
(ii) $a \leq b$ implies $b' \leq a'$,
(iii) $a \vee a' = 1$.

A pair $a, b \in L$ is called *orthogonal* if $a \leq b'$. An ortholattice in which for all $a, b \in L$ if $a \leq b$, then $b = a \vee (a' \wedge b)$ (the orthomodular law) is called an *orthomodular lattice*.

An orthomodular lattice is *separable* if every set of mutually orthogonal elements is at most countable.

A pair a, b of elements of an orthomodular lattice L is called *compatible* if $b = (a \wedge b) \vee (a' \wedge b)$. The set $C(L)$ of all elements of L which are compatible with every element in L is called a *center* of L. If $C(L) = \{0, 1\}$ we say that L is *irreducible*.

An element a of an orthomodular lattice L is called an *atom* if $0 \leq b < a$ implies $b = 0$ for all $b \in L$. If every nonzero element of L contains an atom, then L is called *atomic*. In fact every atomic orthomodular lattice is *atomistic*, i.e. every nonzero element of L is the supremum of all atoms being under it (see [9]).

A lattice is *modular* if for all $a, b, c \in L$ we have $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. An ortholattice which is a modular lattice is called a *modular ortholattice*. It is easy to check that every modular ortholattice is an orthomodular lattice.

A *direct product* $\prod_{\kappa \in H} L_\kappa$ of orthomodular lattices is the orthomodular lattice in which lattice operations and orthocomplementation are defined "coordinatewise".

Let us now introduce measure theoretic notions we shall need in the sequel.

Definition 2.2. A *finite additive measure* on an orthomodular lattice L is a mapping $m : L \to \langle 0, \infty)$ such that

(i) $m(0) = 0$,
(ii) if $a, b \in L$ with $a \leq b'$, then $m(a \vee b) = m(a) + m(b)$.

If moreover $m(1) = 1$, then m is called a *state*.

Definition 2.3. A *valuation* on a lattice L is a mapping $\omega : L \to \langle 0, \infty)$ with properties

(i) $\omega(0) = 0$,
(ii) $\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b)$ for all $a, b \in L$.

We say that a valuation (state) ω is *faithful* if $\omega(a) = 0$ implies $a = 0$ for all $a \in L$.

Finally, let us recall some topological notions. For elements of a poset P the notation $x_\alpha \uparrow x$ means that $x_{\alpha_1} \le x_{\alpha_2}$ whenever $\alpha_1 \le \alpha_2$ in a directed index set $\mathcal{E}$ and $x = \bigvee_\alpha x_\alpha$. The meaning of the symbol $x_\alpha \downarrow x$ is dual. We say that a net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of a poset P *order converges* to a point $x \in P$ (written $x_\alpha \overset{(o)}{\to} x$) if there exist nets $(u_\alpha)_{\alpha\in\mathcal{E}}, (v_\alpha)_{\alpha\in\mathcal{E}} \subseteq P$ such that $u_\alpha \le x_\alpha \le v_\alpha$, $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow x$, $v_\alpha \downarrow x$. If P is a complete lattice, then $x_\alpha \overset{(o)}{\to} x$ iff $x = \bigvee_\beta \bigwedge_{\alpha\ge\beta} x_\alpha = \bigwedge_\beta \bigvee_{\alpha\ge\beta} x_\alpha$. The *order topology* (denoted τ_o) on a lattice L is the finest topology τ on L such that for any net of elements of L order convergence implies τ-convergence.

An orthomodular lattice L is called *order continuous* if for any net $(x_\alpha)_{\alpha\in\mathcal{E}} \subseteq L$ and any $x, y \in L$ we have $x_\alpha \uparrow x$ implies $x_\alpha \wedge y \uparrow x \wedge y$.

Fact 1 ([12])**.** Every complete modular ortholattice is order continuous.

An orthomodular lattice L is called *order topological* if L is order continuous and for elements of L, $x_\alpha \overset{(o)}{\to} x$ iff $x_\alpha \overset{\tau_0}{\to} x$. In such cases lattice operations are continuous.

Fact 2 (Corollary 3.2 of [17])**.** Every complete atomic modular ortholattice is order topological.

A finite additive measure m on an orthomodular lattice L is called *order continuous* if for elements of L, $x_\alpha \overset{(o)}{\to} x$ implies $m(x_\alpha) \to m(x)$.

Fact 3 (Theorems 3.1 and 3.6 from [18])**.** For an order continuous complete atomic orthomodular lattice and an order continuous faithful valuation ω on L it holds $x_\alpha \overset{\tau_o}{\to} x$ iff $\omega(x_\alpha \Delta x) \to 0$, $x_\alpha, x \in L$ (here $x\Delta y = (x \vee y) \wedge (x \wedge y)'$, $x, y \in L$).

Fact 4 ([19], p. 65)**.** For an orthomodular lattice L with a faithful valuation ω there exists a complete orthomodular lattice $\widetilde{L}$ and an order continuous faithful valuation $\tilde\omega$ on $\widetilde{L}$ such that L can be $\tilde\tau_o$-densely embedded into $\widetilde{L}$ (here $\tilde\tau_0$ is the order topology on $\widetilde{L}$) and $\tilde\omega$ extends ω.

Note that the embedding of L into $\widetilde{L}$ in Fact 4 need not be regular, i.e. suprema and infima of infinite subsets of L which exist in L need not be inherited for $\widetilde{L}$. Hence $\widetilde{L}$ need not be the MacNeille completion of L.

We note that in Birkhoff's Lattice Theory [2] at several places of the part about topological lattices should be "order continuous lattice" instead of "topological lattice in its order convergence". For example it should be "Every complete metric lattice is order continuous" instead of the false statement "Every complete metric lattice is a topological lattice in its order convergence". See also remarks in [5], p. 218 and Remark 4.2 in [17], p. 517. Theorem 3.3 of the sequel improves these statements of Birkhoff.

3. MacNeille completion and the existence of a faithful valuation

Let L be a poset. A complete lattice $\widehat{L}$ into which L is embedded is called a *MacNeille completion* of L if L is supremum dense and infimum dense in $\widehat{L}$ (i.e. for every $x \in \widehat{L}$ there are $P, Q \subseteq L$ with $\bigvee P = \bigwedge Q = x$; here we identify L with $\varphi(L)$, where $\varphi : L \to \widehat{L}$ is the embedding). Any two MacNeille completions of a poset are isomorphic (see [20]). A MacNeille completion of an ortholattice L inherits, in a natural way, an orthocomplementation which extends that of L ([3]). We thus regard a MacNeille completion of an ortholattice as an ortholattice.

Theorem 3.1. *The MacNeille completion of an orthomodular lattice with an order continuous faithful valuation is a complete modular and separable ortholattice.*

Proof. Suppose that ω is an order continuous faithful valuation on an orthomodular lattice L. There exists a unique complete orthomodular lattice $\widehat{L}$ into which L can be $\hat{\tau}_0$-densely embedded (here $\hat{\tau}_o$ is the order topology on $\widehat{L}$) and an order continuous faithful valuation $\tilde{\omega}$ on $\widehat{L}$ which extends ω (see [19], p. 65). It follows that $\widehat{L}$ is modular and separable (see [19], p. 36). We identify L with its embedding into $\widehat{L}$. In that sense L is a modular subortholattice of $\widehat{L}$. Assume that $x \in L, z \in \widehat{L}$ and for a net $(x_\alpha)_\alpha \subseteq L$ it holds $x_\alpha \uparrow x$ (in L) and $x_\alpha \uparrow z$ (in $\widehat{L}$). By continuity we have $\hat{\omega}(x_\alpha) = \omega(x_\alpha) \uparrow \omega(x) = \hat{\omega}(x)$ and $\hat{\omega}(x_\alpha) \uparrow \hat{\omega}(z)$, hence $\hat{\omega}(z) = \hat{\omega}(x)$. As evidently $z \le x$ and $\hat{\omega}$ is faithful we conclude that $x = z$. Hence the order continuity of ω implies that all suprema and infima existing in L are inherited for $\widehat{L}$.

To prove that $\widehat{L}$ is a MacNeille completion of L it is enough to show that for every nonzero element $x \in \widehat{L}$ there exists a nonzero element $y \in L$ with $y \le x$. Since $\widehat{L}$ is an order continuous lattice (by Kaplansky's Theorem) and $\hat{\omega}$ is an order continuous faithful valuation on $\widehat{L}$, the order topology $\hat{\tau}_o$ on $\widehat{L}$ coincides with the topology on $\widehat{L}$ induced by the metric

$$\rho_{\hat{\omega}}(x,y) = \hat{\omega}(x \Delta y), \qquad x, y \in \widehat{L}$$

(see [18], Theorems 3.1 and 3.6). Thus for chosen $0 \ne x \in \widehat{L}$ there exists a sequence $(x_n)_{n=1}^\infty \subseteq L$ with $x_n \xrightarrow{\hat{\tau}_0} x$ which is equivalent to $\hat{\omega}(x_n \Delta x) \to 0$. Let us denote $V_k = \{y \in \widehat{L} | \hat{\omega}(y \Delta x) < \frac{1}{2^k}\}$ and $\overline{V}_k = \{y \in \widehat{L} | \hat{\omega}(y \Delta x) \le \frac{1}{2^k}\}$, $k \in N$. According to $\hat{\omega}(x_n \Delta x) \to 0$, for every $k \in N$ there exists $n_k \in N$ such that $x_{n_k} \in V_k$. As $\hat{\omega}$ is a valuation we have

$$\hat{\omega}((x_{n_k} \vee x_{n_{k+1}}) \Delta x) \le \hat{\omega}(x_{n_k} \Delta x) + \hat{\omega}(x_{n_{k+1}} \Delta x) < \frac{1}{2^k} + \frac{1}{2^{k+1}} < \frac{1}{2^{k-1}}.$$

Similarly

$$\hat{\omega}((x_{n_k} \vee x_{n_{k+1}} \vee x_{n_{k+2}}) \Delta x) \le \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} < \frac{1}{2^{k-1}}$$

and also

$$\hat{\omega}\left(\left(\bigvee_{l=k}^{k+p} x_{n_l}\right) \Delta x\right) < \frac{1}{2^{k-1}} \quad \text{for every } p \in N.$$

Hence $\bigvee_{l=k}^{k+p} x_{n_l} \in \overline{V}_{k-1}$ for every $p \in N$. This implies that $\bigvee_{l=k}^{\infty} x_{n_l} \in \overline{V}_{k-1}$, as for every $k \in N$ the sets $\overline{V}_k$ are closed in $\hat{\tau}_0$. It follows that $\bigwedge_{k=1}^{\infty} \bigvee_{l=k}^{\infty} x_{n_l} \in \bigcap_{k=1}^{\infty} \overline{V}_k = \{x\}$.

By similar arguments, as we also have $x'_n \xrightarrow{\hat{\tau}_0} x'$, implies there exists a subsequence $(x'_{m_l})_{l=1}^\infty$ of the sequence $(x'_n)_{n=1}^\infty$ such that $\bigwedge_{k=1}^\infty \bigvee_{l=k}^\infty x'_{m_l} = x'$, and consequently $x = \bigvee_{k=1}^\infty \bigwedge_{l=k}^\infty x_{m_l}$. Let $k_0 \in L$. Then either there exists $0 \ne y \in L$ such that $y \le x_{m_l}$ for every $l \ge k_0$ and consequently $y \le x$, or $\bigwedge_{l=k_0}^\infty x_{m_l} = 0$ in L and hence also in $\widehat{L}$. If for all $k \in N$ we have $\bigwedge_{l=k}^\infty x_{m_l} = 0$, then $x = 0$, which contradicts the assumption. Hence there exists $0 \ne y \in L$ with $y \le x$.

If a complete modular ortholattice L is atomic and irreducible, then every chain in L is finite. It follows that every element of L is a join of finitely many pairwise orthogonal atoms. Moreover, there exists the unique natural number n for L, namely the number of all atoms in a block. Thus every block of L is isomorphic to a power set of n atoms.

Theorem 3.2. *For a complete atomic modular ortholattice L the following conditions are equivalent.*

(i) *The center $C(L)$ of L is separable.*
(ii) *L is separable.*
(iii) *There exists a faithful state on L.*
(iv) *There exists a faithful valuation on L.*
(v) *There exists an order continuous faithful valuation on L.*
(vi) *L is isomorphic to a direct product of at most countably many irreducible, separable, complete, atomic, modular ortholattices.*

In such cases for every order continuous faithful valuation ω on L the metric convergence induced by ω coincides with the order convergence on L, i.e. for any $(x_\alpha)_\alpha \subseteq L$ and $x \in L$

$$x_\alpha \overset{(o)}{\to} x \quad \textit{iff} \quad \omega(x_\alpha \Delta x) \to 0.$$

Proof. Clearly (v)⇒(iv)⇒(iii) and (ii)⇒(i).

(iii)⇒(ii): Suppose that μ is a faithful state on L and $P \subseteq L$ is an uncountable set of mutually orthogonal elements of L. Then there exist natural numbers n_1, n_2 such that the set $\{x \in P | \mu(x) \geq \frac{1}{n_1}\}$ is uncountable and $n_2 > n_1\mu(1)$. It follows that there is a set $\{x_1, x_2, \ldots, x_{n_k}\} \subseteq \{x \in P | \mu(x) \geq \frac{1}{n_1}\}$ and $\mu(1) < n_2 \cdot \frac{1}{n_1} \leq \sum_{k=1}^{n_2} \mu(x_k) \leq \mu(1)$ which is a contradiction. Thus L is separable.

(vi)⇒(iv): Suppose that L is isomorphic to a direct product $\prod_{k\in J} L_k$, where L_k are irreducible separable complete modular atomic ortholattices and J is at most countable. It is known that on every L_k, $k \in J$, there exists a faithful valuation with range $\{m/n_k | 0 \leq m \leq n_k, m \in N\}$. Here n_k is a number (unique for L_k) of atoms in a block of L_k and for $x \in L_k$, $\omega_k(x) = m/n_k$ if x is a join of m mutually orthogonal atoms (see [9], p. 198). Let α_k for $k \in J$ be positive real numbers with $\sum_{k\in J} \alpha_k = 1$. Then the mapping $\omega : \prod_{k\in J} L_k \to \langle 0, \infty)$ defined by the formula

$$\omega((x_k)_{k\in J}) = \sum_{k\in J} \alpha_k \omega_k(x_k), \qquad (x_k)_{k\in J} \in \prod_{k\in J} L_k$$

is a faithful valuation on $\prod_{k\in J} L_k$.

(i)⇒(vi): Since L is complete and supremum and infimum of an arbitrary subset of $C(L)$ belongs to $C(L)$ ($C(L)$ is actually a complete Boolean subalgebra of L, see [9], pp. 24 and 26) we have for any net $(x_\alpha)_\alpha \subseteq C(L)$

$$x_\alpha \overset{(o)}{\to} x \text{ (in } L) \quad \text{iff} \quad x \in C(L) \text{ and } x_\alpha \overset{(o)}{\to} x \text{ in } C(L)).$$

Applying the fact that a subset D of a poset P is a closed set in the order topology on P iff the set D contains limits of all of its order convergent nets, we obtain that $C(L)$ is a closed set in the order topology on L. Thus $\tau_{OL} \cap C(L) = \tau_{OC(L)}$, where τ_{OL} and $\tau_{OC(L)}$ denote the order topologies (i.e. the families of all open subsets) on L and $C(L)$ respectively. By [17] every complete atomic modular ortholattice is order topological. Thus L is order topological and by the previous remark on the order convergence and order topology on $C(L)$ we obtain that $C(L)$ is also order-topological. This implies, applying Theorem 2.2 from [5], that $C(L)$ is atomic. As $C(L)$ is separable the set of all atoms of $C(L)$ is at most countable. Let us denote $A = \{p_k | k \in J\}$ the set of atoms of $C(L)$. Since atoms of $C(L)$ are mutually orthogonal with $\bigvee\{p_k | k \in J\} = 1$ we obtain that L is isomorphic to the direct product $\prod_{k\in J}[0, p_k]$ (see [9]). This isomorphism φ is defined by $\varphi(x) = (x_k)_{k\in J}$, where $x_k = x \wedge p_k$ for $k \in J, x \in L$.

(iv)⇒(v): By Kaplansky's theorem L is order continuous. We define the map $\nu : L \to \langle 0, \infty)$ as follows: For $0 \neq x \in L$ we put $\nu(x) = \sup\{\omega(y) | y$ is the join of a finite set of atoms under $x\}$. Moreover $\nu(0) = 0$. By Theorem 3.2 of [16] ν is an order continuous finite additive measure on L. Evidently $\nu(y) = \omega(y)$ for every $y \in L$ which is the join of a finite set of atoms under x. Let us show that ν is a valuation. Suppose that $x, y \in L$ and $A_x = \{p \in L | p$ is an atom with $p \leq x\}$, $A_y = \{q \in L | q$ is an atom with $q \leq y\}$. For every finite set $\alpha \subseteq A_x \cup A_y$ with $\alpha \cap A_x \neq \varnothing \neq \alpha \cap A_y$ we put $x_\alpha = \bigvee \alpha \cap A_x$, $y_\alpha = \bigvee \alpha \cap A_y$. Then $x_\alpha \uparrow x$, $y_\alpha \uparrow y$ and also $x_\alpha \vee y_\alpha \uparrow x \vee y, x_\alpha \wedge y_\alpha \uparrow x \wedge y$. Therefore ν is order continuous, we obtain $\nu(x \vee y) = \lim_\alpha \nu(x_\alpha \vee y_\alpha) = \lim_\alpha \omega(x_\alpha \vee y_\alpha) = \lim_\alpha[\omega(x_\alpha) + \omega(y_\alpha) - \omega(x_\alpha \wedge y_\alpha)] = \nu(x) + \nu(y) - \nu(x \wedge y)$. Hence ν is an order continuous faithful valuation on L.

The rest of the theorem follows by the statement that for an order continuous faithful valuation ω on L we have $x_\alpha \overset{\tau_{OL}}{\to} x$ iff $\omega(x_\alpha \Delta x) \to 0$, $x_\alpha, x \in L$ (see [18], Theorems 3.1 and 3.6). Since L is order topological we obtain $x_\alpha \overset{(o)}{\to} x$ iff $\omega(x_\alpha \Delta x) \to 0$, $x_\alpha, x \in L$.

Remark. Note that a construction of an order continuous faithful valuation in Theorem 3.2 is given in parts (vi)⇒(iv) and (iv)⇒(v) of the proof, using the isomorphism φ from the end of the part (i)⇒(vi).

Actually, Theorem 3.2 together with the statement that an order topological complete modular ortholattice is atomic (see [5], Theorem 2.2) gives the following result:

Theorem 3.3. *For a complete separable modular ortholattice L the following conditions are equivalent*:

(i) *There exists a faithful valuation ω on L such that for any $(x_\alpha)_\alpha \subset L$, $x \in L$ $x_\alpha \overset{(o)}{\to} x$ iff $\omega(x_\alpha \Delta x) \to 0$,*
(ii) *L is atomic,*
(iii) *L is order topological.*

Recall that an atomistic ortholattice is called *strongly compactly atomistic* if for every set M of atoms of L and every atom p of L with the property $p \leq x$, for every upper bound x of the set M, there exists a finite set $F \subseteq M$ with $p \leq \bigvee F$.

Theorem 3.4. *For a modular atomic ortholattice the following conditions are equivalent*:

(i) *There exists an order continuous faithful valuation ω on L.*
(ii) *L is strongly compactly atomistic and separable.*
(iii) *L is separable and the MacNeille completion of L is modular.*
(iv) *L is isomorphic to a supremum (infimum) dense subset of a direct product of at most countably many irreducible, complete, modular ortholattices.*

Proof. (ii)⇒(iii) This follows by Theorem 3.5 from [17].

(iii)⇒(i): Let us denote by $\widehat{L}$ the MacNeille completion of L. Since L is supremum dense in $\widehat{L}$ we obtain that $\widehat{L}$ is atomic and it has the same set of all atoms as L. Consequently $\widehat{L}$ is separable. In view of Theorem 3.2 there exists an order continuous faithful valuation $\hat{\omega}$ on $\widehat{L}$. Since all suprema and infima existing in L are inherited for $\widehat{L}$ we conclude that the restriction of $\hat{\omega}$ to L is an order continuous faithful valuation on L.

(i)⇒(iii): This follows by Theorem 3.1.

(iii)⇔(iv): By Theorem 3.2, using that L is atomic and separable if and only if the MacNeille completion of L is atomic and separable.

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