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QUOTIENT DIVISIBLE ABELIAN GROUPS

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Dedicated to the memory of Ross A. Beaumont

ABSTRACT. An abelian group G is called quotient divisible if G is of finite torsion-free rank and there exists a free subgroup $F \subset G$ such that G/F is divisible. The class of quotient divisible groups contains the torsion-free finite rank quotient divisible groups introduced by Beaumont and Pierce and essentially contains the class $\mathcal{G}$ of self-small mixed groups which has recently been investigated by several authors. We construct a duality from the category of quotient divisible groups and quasi-homomorphisms to the category of torsion-free finite rank groups and quasi-homomorphisms. Our duality when restricted to torsion-free quotient divisible groups coincides with the duality of Arnold and when restricted to $\mathcal{G}$ coincides with the duality previously constructed by the authors.

All groups will be additive and abelian. The torsion-free rank of a group G is the rank of $G/T(G)$, where $T(G)$ is the torsion subgroup of G. The p-rank of G is the rank of G/pG. For a group G and prime p we let $\widehat{G_p}$ be the p-adic completion of $G/p^\omega G$ and regard $G/p^\omega G \subset \widehat{G_p}$. A $\diamond$ marks the end of a proof.

Definition 1. An abelian group G is **quotient divisible** (qd) if G is of finite torsion-free rank and there exists a free subgroup $F \subset G$ with G/F a divisible group.

We remark that the torsion-free qd groups are precisely the classical quotient divisible groups introduced by Beaumont and Pierce in 1961 [BP]. The following proposition is elementary.

Proposition 2. *For an abelian group G of finite torsion-free rank the following are equivalent:*
(1) *G is qd.*
(2) *There exists a free subgroup $F \subset G$ such that, for each positive integer m, $G = F + mG$.*
(3) *There exists a free subgroup $F \subset G$ such that, for each prime p, $G = F + pG$.*

Definition 3. For each prime p let $M(p)$ be a module over the ring of p-adic integers $\widehat{Z_p}$. A subgroup $A \subset \prod_p M(p)$ **satisfies the projection condition** if there is a free subgroup of finite rank $F \subset A$ such that, for every p, the natural projection of F into $M(p)$ generates $M(p)$ as a $\widehat{Z_p}$- module.

We call a free subgroup F of a group G **full** if G/F is torsion.

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Theorem 4. *Let A be reduced of torsion-free rank n. Then A is qd if and only if A is (can be embedded as) a pure subgroup of a direct product of p-adic modules $A \subset \prod_p M(p)$ such that A satisfies the projection condition.*

Proof. Suppose A is reduced qd of torsion-free rank n. Let $F \subset A$ be free of rank n with A/F divisible. By Proposition 2, if T_p is the p-torsion part of A, then T_p cannot have $n+1$ nonzero cyclic summands. Hence each T_p is finite. For each p write $A = T_p \oplus A'_p$ where A'_p has no p-torsion. The first Ulm subgroup $A^1 = \bigcap_p p^\omega A$ is contained in $\bigcap_p A'_p$ since an element of infinite p-height cannot have a nonzero T_p component in any of the previous decompositions. We claim that for each $a \in A^1$ and prime p, there exists a $b \in A^1$ such that $a = pb$. Plainly, we can choose $b \in A'_p$ with $a = pb$. If $q \neq p$ is prime then b will have infinite q-height since a does. For each positive integer k there exists $b_k \in A'_p$ with $p^{k+1}b_k = a$. Hence $p(b - p^k b_k) = 0$ and, since $(b - p^k b_k) \in A'_p, (b - p^k b_k) = 0$. Thus b has infinite p-height so that $b \in A^1$. We have shown that A^1 is divisible. Since A is reduced, $A^1 = 0$.

It follows that A is embedded as a pure subgroup of its Z-adic completion $A\hat{}= \prod_p M(p)$ where $M(p) = A\hat{_p}$, the p-adic completion of $A/p^\omega A$. (See [Fu], Sections 39 and 40.) It remains to show that A satisfies the projection condition. In our context each projection π_p is the factor map $A \to A/p^\omega A$ followed by inclusion $A/p^\omega A \to M(p)$. Thus each $M(p)/\pi_p(A)$ is divisible. But A/F is divisible so that $\pi_p(A)/\pi_p(F)$ is also divisible. Hence $M(p)/\pi_p(F)$ is divisible. Since $Z\hat{_p}[\pi_p(F)]$ is a finitely generated submodule, hence a summand, of the p-adic module $M(p)$, and $M(p)$ is reduced, it follows that $M(p) = Z\hat{_p}[\pi_p(F)]$ for each prime p.

Conversely, suppose that A is a pure subgroup of $\prod_p M(p)$ such that A satisfies the projection condition. Let $F = \bigoplus Zx_j \subset A$ be a full free subgroup such that $M(p) = Z\hat{_p}[\pi_p(F)]$ for each prime p. Let $a \in A$ and q be a fixed prime. Then $\pi_q(a) = \sum \alpha_j \pi_q x_j$ for q-adic integers α_j. For each j let $a_j \in Z$ be congruent to α_j mod q. Then $a - \sum a_j x_j$ is divisible by q in $\prod_p M(p)$, hence divisible by q in A. We have shown that A/F is q-divisible for each q and the proof is complete. ◇

Definition 5. Let $\mathcal{QD}$ be the category with objects qd groups with reduced torsion part and maps quasi-homomorphisms; that is,

$$Hom_{\mathcal{QD}}(A, A') = Q \otimes_Z Hom_Z(A, A').$$

Let $\mathcal{QD}_0$ be the full subcategory of $\mathcal{QD}$ with objects torsion-free quotient divisible groups. Let $\mathcal{QTF}$ be the category with objects torsion-free finite rank groups and maps quasi-homomorphisms. Let $\mathcal{QG}$ be the category with objects self-small mixed abelian groups G such that $G/T(G)$ is finite rank divisible and maps quasi-homomorphisms.

The category $\mathcal{QG}$ has been investigated by several authors ([AGW], [FiW], [FoW], [GW], [VW], [Wi]). The relation between $\mathcal{QG}$ and $\mathcal{QD}$ is described in the following proposition.

Proposition 6. *The category $\mathcal{QG}$ can be identified with the full subcategory $\mathcal{QD}_t$ of $\mathcal{QD}$ whose objects H satisfy the additional restriction that the p-adic modules $M(p)$ of Theorem 4, defined with respect to $A = H/V, V$ the maximal divisible subgroup of H, are torsion for each p.*

Proof. It follows from Corollary 2.4 of [AGW] that every reduced group A in $\mathcal{QG}$ is a pure subgroup of $\prod T_p(A)$, where $T_p(A)$ is the p-torsion subgroup of A, with the projection condition holding for some full free subgroup $F \subset A$ for almost all primes p. Let S be the finite subset of primes such that $Z_p^\frown[\pi_p(F)] \neq T_p(A)$. For $p \notin S$, $T_p(A)$ will be finite of rank less than or equal to rank F = torsion-free rank A. In order for A to be self-small the groups $T_p(A), p \in S$, also must necessarily be finite [AGW]. Thus any object G of $\mathcal{QG}$ is a direct sum $G = V \oplus A = V \oplus (B \oplus A')$ where V is finite rank torsion-free divisible, $B = \bigoplus_{p \in S} T_p(A)$ is a finite group and $A' = A \cap \prod_{p \notin S} T_p(A)$ is a pure subgroup of $\prod T_p(A')$ with the projection condition. By Theorem 4, A' is in $\mathcal{QD}$. Hence each object G of $\mathcal{QG}$ is a direct sum $G = B \oplus (V \oplus A')$ of a finite group and an object $H = V \oplus A'$ of $\mathcal{QD}$. Since we are working in quasi-homomorphism categories the correspondence $G \to H$ is an embedding of $\mathcal{QG}$ as a full subcategory of $\mathcal{QD}_t$.

To show that the embedded copy of $\mathcal{QG}$ in $\mathcal{QD}$ is exactly $\mathcal{QD}_t$, let A be a pure subgroup of a product of torsion p-adic modules $\prod_p M(p)$ with the projection condition. We show that, for each fixed prime q, $M(q) = T_q(A)$ and, hence, by Corollary 2.4 of [AGW], that A is a group in $\mathcal{QG}$. Let q be fixed and $m \in M(q)$. There is an $f = (f_p) \in F$, a full free subgroup of A, with $f_q = m$. If $q^k M(q) = 0$ ($M(q)$ is finitely generated) then $q^k f \in \prod_{p \neq q} M(p) \cap A$. Since elements of $\prod_{p \neq q} M(p)$ have infinite q-height and A is pure in $\prod_p M(p)$ it follows that $q^k f = q^{2k} a$ for some $a \in A$. Thus $q^k(f - m) = q^{2k} a = q^k(q^k a)$. Since division by q in $\prod_{p \neq q} M(p)$ is unique we have $f - m = q^k a \in A$. Thus $m \in A$ and $M(q)$ coincides with the q-torsion subgroup of A. The proof is complete. $\diamond$

For the reader's convenience, we summarize constructions that have been useful in the study of torsion-free finite rank groups. For additional details see [Fo1]. Let G be torsion-free of rank n with free subgroup $F = \bigoplus_{i=1}^n Zx_i$. For each p let r_p be the p-rank of G. Then the torsion group G/F has the form:

$$G/F = \bigoplus_p [G/F]_p = \bigoplus_p [\bigoplus_{i=1}^{r_p} Z(p^{k_{ip}}) \oplus \bigoplus_{i=r_p+1}^{n} Z(p^\infty)], 0 \leq k_{ip} < \infty. \tag{1}$$

Fix a prime p. For each of the first r_p cyclic summands in (1) there is a collection of integers $a_{ij}^p, 1 \leq j \leq n$, such that for $y_i^p = p^{-k_{ip}} \sum_{j=1}^n a_{ij}^p x_j$, $y_i^p + F$ is a generator of that summand. If $\alpha_{ij}^p = a_{ij}^p + p^{k_{ip}} Z$ then, for $1 \leq i \leq r_p$, we obtain a relation $\sum_{j=1}^n \alpha_{ij}^p x_j = 0$ in $G/p^{k_{ip}}G$. Here we regard each $G/p^{k_{ip}}G$ as a $Z/p^{k_{ip}}Z$ module in the natural way and abuse notation by writing x_j for $x_j + p^{k_{ip}}G$. (If $k_{ip} = 0$ we choose the zero relation $\sum_{j=1}^n 0x_j = 0$.)

For each of the $n - r_p$ quasi-cyclic summands in (1) we choose a set of generators $\{y_i^p(k) + F : 1 \leq k < \infty\}$ of that summand with $y_i^p(1) + F$ of order p and, for $k \geq 2$, $p[y_i^p(k) + F] = y_i^p(k-1) + F$. Exactly as in the previous paragraph, each $y_i^p(k)$ determines a relation $\sum_{j=1}^n \alpha_{ij}^p(k) x_j = 0$ in $G/p^k G$. For each fixed j the sequence $\langle \alpha_{ij}^p(k) \rangle_k$ determines a p-adic integer α_{ij}^p. We obtain, for each i with $r_p < i \leq n$, a p-adic relation $\sum_{j=1}^n \alpha_{ij}^p x_j = 0$ in $G_p^\frown$. For p-adic relations we continue our abuse of notation by simply writing x_j for $(x_j + p^\omega G) \in G/p^\omega G \subset G_p^\frown$. Note that if $r_p = 0$ for some p, then $[G/F]_p$ is divisible. In this case we obtain no finite p-relations, $G_p^\frown = 0$, and the p-adic relations can be taken to be $1x_j = 0, 1 \leq j \leq n$. If $r_p = n$ then $[G/F]_p$ has no quasi-cyclic summands and we obtain no p-adic relations.

For each p, we write this set of p-relations in matrix form: $M_G^p X = 0$, where X is the $n \times 1$ column vector with entries $x_1, ..., x_n$ and, for each p, M_G^p is an $n \times n$

matrix with the i-th row consisting of elements of the ring $Z/p^{k_{ip}}Z$ for $i \le r_p$ and elements of the ring $Z_p^{\frown}$ for $r_p < i \le n$. In this way, to each group G and fixed full free subgroup F, we associate a set of matrices $\{M_G^p\}$.

Conversely, given a set of $n \times n$ matrices $\{M^p\}$ indexed by the primes such each row of each M^p consists of elements of the same ring, either a residue class ring $Z/p^{k_{ip}}Z$ or $Z_p^{\frown}$, we can invert our construction to obtain a torsion-free group of rank n. The group is constructed by starting with an abstract free group $F = \bigoplus_{i=1}^n Zu_i$ and adjoining generators corresponding to the relations $M^p X = 0$. For any G, the group determined by the collection of matrices $\{M_G^p\}$ will be exactly G. In the proof of our main theorem we will explicitly detail the inverse construction.

We first prove two lemmas that will be needed in the proof of our main theorem.

Lemma 7. *Let G and H be torsion-free finite rank groups, $\{x_1, ..., x_n\}$ a maximal linearly independent set in G and $\{y_1, ..., y_n\}$ an arbitrary set of elements of H. Then there exists a homomorphism $f : G \to H$ with $f(x_i) = y_i, 1 \le i \le n$, if and only if* $M_G^p \begin{pmatrix} y_1 \\ \cdot\cdot \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot\cdot \\ 0 \end{pmatrix}$ *for all primes p.*

Proof. Let f be the mapping from the divisible hull of G to the divisible hull of H defined by $f(x_i) = y_i, 1 \le i \le n$. The requirement that all generators of G will be carried by f into H is equivalent to the requirement that all of the above matrix equalities hold. ◇

Definition 8. For $0 \le k < \infty$, a p^k-relation of a torsion-free finite rank group G with maximal independent set $\{x_1, ..., x_n\}$ is an element $(\alpha_1, ..., \alpha_n) \in (Z/p^k Z)^n$ such that $\sum_{i=1}^n \alpha_i x_i = 0$ in $G/p^k G$. A p^∞-relation of G is an element $(\alpha_1, ..., \alpha_n) \in (Z_p^{\frown})^n$ such that $\sum_{i=1}^n \alpha_i x_i = 0$ holds in $G_p^{\frown}$. (We continue to write x_i for the appropriate coset of x_i.)

Every $Z_p^{\frown}$-homomorphism from $[G/F]_p = [\bigoplus_{j=1}^{r_p} Z/p^{k_{jp}}Z] \oplus [\bigoplus_{j=r_p+1}^n (Z(p^\infty))_j]$ to $Z/p^k Z$, $0 \le k < \infty$, or to $Z(p^\infty)$ can be represented by a row of homomorphisms $\varphi = (\varphi_1, ..., \varphi_n)$. We can write the elements a of

$$[\bigoplus_{j=1}^{r_p} Z/p^{k_{jp}}Z] \oplus [\bigoplus_{j=r_p+1}^{n} (Z(p^\infty))_j]$$

as $n \times 1$ column matrices and thereby represent $\varphi(a)$ as a matrix product.

Lemma 9. *With notation as above, an element $(\alpha_1, ..., \alpha_n)$ of $(Z/p^k Z)^n$ or $(Z_p^{\frown})^n$ is a p^k-relation or a p^∞-relation if and only if $(\varphi_1, ..., \varphi_n)M_G^p = (\alpha_1, ..., \alpha_n)$ for some $(\varphi_1, ..., \varphi_n)$. Furthermore $(\varphi_1, ..., \varphi_n)M_G^p = (0, ..., 0)$ only if $(\varphi_1, ..., \varphi_n) = (0, ..., 0)$.*

Proof. These results follow from the fact that the rows of each M_G^p are the relations obtained from generators of G/F which are linearly independent. For details see [Fo1], Lemma 2. ◇

Theorem 10. *The categories $\mathcal{QD}$ and $\mathcal{QTF}$ are dual.*

Proof. We construct a contravariant equivalence d from $\mathcal{QD}$ to $\mathcal{QTF}$. Let G be an object of $\mathcal{QD}$. Then G is a qd group such that its maximal divisible subgroup D is torsion-free. Hence $A = G/D$ is a reduced qd group which, by Theorem 4, can be regarded as a pure subgroup $A \subset A^{\frown} = \prod_p A_p^{\frown}$ with the projection condition. Let $F = \bigoplus_{i=1}^n Zx_i$ be a full free subgroup such that $G/\bigoplus_{i=1}^n Zx_i$ is divisible. For each prime p, let π_p: $G \to A_p^{\frown}$ be the natural map $G \to A$ followed by projection of A into $A_p^{\frown}$.

Because of the projection condition each $A_p^{\frown}$ is generated over $Z_p^{\frown}$ by the set $\{\pi_p(x_1), ..., \pi_p(x_n)\}$. It follows that $A_p^{\frown} \cong [\bigoplus_{j=1}^{s_p} Z/p^{k_{jp}}Z] \oplus [\bigoplus_{j=s_p+1}^n (Z_p^{\frown})_j]$. Here, for each p, the s_p are integers with $0 \le s_p \le n$ and the $k_{jp}, 1 \le j \le n$, are a non-decreasing sequence of nonnegative integers. For each fixed i we have $\pi_p(x_i) = (\alpha_{i1}, ..., \alpha_{in})$ where the first s_p entries are in the appropriate residue class rings $Z/p^{k_{jp}}\ Z$ and the remaining entries are p-adic integers We obtain, for each p, an $n \times n$ matrix $M_p^G = M(p, G, \{x_1, ..., x_n\})$ whose i-th row is $\pi_p(x_i)$. For $j \le s_p$ the j-th column of M_p^G will contain elements of the ring $Z/p^{k_{jp}}\ Z$. The remaining columns of M_p^G will contain p-adic integers. Of course, just as in the construction of M_G^p, we allow the possibilities $s_p = 0$ ($A_p^{\frown}$ has no finite part) and $s_p = n$ ($A_p^{\frown}$ has no torsion-free part) as well as the possibility that some or all $k_{jp} = 0$.

We first construct the object dG. Let QG be the divisible hull of $G/T(G)$. We identify QG with $\bigoplus_{i=1}^n Qx_i$. Let $\{x_1^*, ..., x_n^*\}$ be the basis of $QG^* = Hom(QG, Q)$ dual to $\{x_1, ..., x_n\}$. Then dG is the subgroup of $Hom(QG, Q)$ that is generated by $\bigoplus_{i=1}^n Zx_i^*$ and the following set. For each prime p and $1 \le j \le s_p$ we include the generator $(a_{1j}x_1^* + ... + a_{nj}x_n^*)/p^{k_{jp}}$. Here the a_{ij} are integers such that $\alpha_{ij} = a_{ij} + p^{k_{jp}}Z$, $1 \le i \le n$, is the j-th column of M_p^G. For each p-adic column $(\alpha_{1j}, ..., \alpha_{nj})$ of M_p^G $(j > s_p)$ and for each positive integer k, we include a generator $(a_{1j}(k)x_1^* + ... + a_{nj}(k)x_n^*)/p^k$ where $a_{ij}(k)$ is an integer congruent to α_{ij} modulo p^k.

Next, let $G, H \in \mathcal{QD}$ and $f : G \to H$ be a quasi-homomorphism. In particular f can be regarded as a map from QG to QH. The standard dual transformation $f^* = Hom(f, Q)$ maps QH^* to QG^*. Put $df = f^* \mid_{dH}$.

We prove that df is a quasi-homomorphism from dH to dG. Let $\{y_1, ..., y_m\}$ be the maximal independent subset of H used in the construction of dH and let L be the $n \times m$ rational matrix defined by the equation $\begin{pmatrix} f(x_1) \\ \cdots \\ f(x_n) \end{pmatrix} = L \begin{pmatrix} y_1 \\ \cdot\cdot \\ y_m \end{pmatrix}$.

If $g \in G$ has coordinate row $(q_1, ..., q_n)$ with respect to $\{x_1, ..., x_n\}$ then $f(g)$ has coordinate row $(q_1, ..., q_n)L$ with respect to $\{y_1, ..., y_m\}$.

By replacing f with a suitable integral multiple of f we can assume that f is a homomorphism and that L has integral entries. We show that for this new f, df will be a homomorphism from dH to dG. Let B be H factored by its maximal divisible subgroup. Then f induces a homomorphism from A to B and hence a $Z_p^{\frown}$-homomorphism $f_p^{\frown}$: $A_p^{\frown} \to B_p^{\frown}$ for each p. Write

$$A_p^{\frown} = [\bigoplus_{j=1}^{s_p} (Z/p^{k_{jp}}Z)e_j] \oplus [\bigoplus_{j=s_p+1}^{n} (Z_p^{\frown})e_j],$$

$$B_p^{\frown} = [\bigoplus_{j=1}^{s'_p} (Z/p^{k'_{jp}}Z)e'_j] \oplus [\bigoplus_{j=s'_p+1}^{m} (Z_p^{\frown})e'_j].$$

Here, for $1 \le j \le s_p$ (resp.$1 \le j \le s'_p$), order $e_j = p^{k_{jp}}$ (resp. order $e'_j = p^{k'_{jp}}$), and the e_j (resp. e'_j) are of infinite order for $j > s_p$ (resp. $j > s'_p$). Note that some or all of the e_j or e'_j and hence the accompanying coefficient ring $Z/p^{k_{jp}}Z$ or $Z/p^{k'_{jp}}Z$ can be zero.

Let Δ_p be the matrix defined by the equation $\begin{pmatrix} f_p\widehat{\ }(e_1) \\ \cdots \\ f_p\widehat{\ }(e_n) \end{pmatrix} = \Delta_p \begin{pmatrix} e'_1 \\ \cdots \\ e'_m \end{pmatrix}$. If $a \in A_p\widehat{\ }$ has coordinate row $(\alpha_1, ..., \alpha_n)$ with respect to the decomposition of $A_p\widehat{\ }$ then $f_p\widehat{\ }(a)$ has coordinate row $(\alpha_1, ..., \alpha_n)\Delta_p$ with respect to the decomposition of $B_p\widehat{\ }$.

Let π'_p: $H \to B_p\widehat{\ }$ be the map employed in the definition of the matrix M_p^H, exactly as π_p was used to define M_p^G. Then $\pi'_p f = f_p\widehat{\ }\pi_p$ for each p. It is a simple exercise in linear algebra to verify the matrix equalities $LM_p^H = M_p^G\Delta_p$. By construction of dG, $(x_1^*, ..., x_n^*)M_p^G = (0, ..., 0)$ for each prime p. It follows that $(x_1^*, ..., x_n^*)LM_p^H = (0, ..., 0)$ for each p. Since f^* is the dual of f, $\begin{pmatrix} f^*(y_1^*) \\ \cdots \\ f^*(y_m^*) \end{pmatrix} = L^t \begin{pmatrix} x_1^* \\ \cdots \\ x_n^* \end{pmatrix}$ where L^t is the transpose of L. Transposing this matrix equation gives $(f^*(y_1^*),...,f^*(y_m^*)) = (x_1^*, ..., x_n^*)L$. Thus $(f^*(y_1^*),...,f^*(y_m^*))M_p^H = (0, ..., 0)$ for each p. By construction of dH the matrix M_p^H is the transpose of the matrix M_{dH}^p. Thus, we can appply Lemma 7 to conclude that f^* restricted to dH is a homomorphism into dG. We have shown that df is a homomorphism from dH to dG, as desired.

Let G be a torsion-free finite rank group with maximal independent set $\{x_1, ..., x_n\}$. To construct the inverse functor d' for d we start with the associated set of matrices $\{M_G^p\}$. Then we construct a quotient divisible group $d'G$ such that the divisible hull of $d'G/T(d'G)$ is equal to $Hom(QG, Q)$ and such that $M_p^{d'G}$ is equal to the transpose of the matrix M_G^p for each p.

More precisely, given G and $\{x_1, ..., x_n\}$ consider the columns of the matrix M_G^p . Each of these columns can be regarded as an element of the $Z_p\widehat{\ }$-module $M(p) = [\bigoplus_{j=1}^{r_p}(Z/p^{k_{jp}}Z)] \oplus [\bigoplus_{j=r_p+1}^{n}(Z_p\widehat{\ })_j]$. If $M(p)$ is not generated over $Z_p\widehat{\ }$ by the columns of M_G^p, we can obtain a nonzero $Z_p\widehat{\ }$-module homomorphism $\varphi = (\varphi_1, ..., \varphi_n)$: $M(p) \to Z/pZ$ mapping all the columns to zero. But $(\varphi_1, ..., \varphi_n)M_G^p = (0, ..., 0)$, contradicting Lemma 9. For $1 \le i \le n$, let $v_i = (v_{ip})$ be the element of $\prod_p M(p)$ such that v_{ip} is the i-th column of M_G^p. Let A be the pure subgroup of $\prod_p M(p)$ generated by the torsion subgroup of $\prod_p M(p)$ and the set $\{v_1, ..., v_n\}$. If M_G^p is the zero matrix for all p ($G = \bigoplus Zx_i$) then $A = 0$.

Let $F^* = \bigoplus_{i=1}^n Zx_i^* \subset G^* = Hom(QG, Q)$. The assignment $x_i^* \to v_i, 1 \le i \le n$, determines a map $g : Hom(QG, Q) \to Q \otimes A$. Let $d'G = A \oplus \ker(g)$. Since $\{v_{ip} : 1 \le i \le n\}$ generates $M(p)$ over $Z_p\widehat{\ }$ for each p, then $d'G$ satisfies the projection condition with respect to $F = \bigoplus Zx'_i$, where $x'_1, ..., x'_n$ are chosen such that $x'_i = v_i + d_i, d_i \in \ker(g)$. Thus $d'G$ is an object of $\mathcal{QD}$.

Let $f : G \to H$ be a homomorphism of torsion-free finite rank groups. We need to construct a quasi-homomorphism $d'f : d'H \to d'G$. Let $f^* : H^* \to G^*$ be the standard dual map. Since, for any torsion-free finite rank G, we can identify G^* with $Q \otimes d'G$ via $x_i^* \to 1 \otimes x'_i$, we regard f^* as a map from $Q \otimes d'H$ to $Q \otimes d'G$. We put $d'f = f^*$ and claim that

$$d'f = f^* \in Q \otimes Hom(d'H, d'G).$$

As before, we can assume the matrix L of f with respect to the distinguished bases $\{x_1, ..., x_n\}$ for G and $\{y_1, ..., y_m\}$ for H has integral entries. By Lemma 7

$M_G^p \begin{pmatrix} f(x_1) \\ \cdots \\ f(x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot\cdot \\ 0 \end{pmatrix}$ for all p. It follows that $M_G^p L \begin{pmatrix} y_1 \\ \cdot\cdot \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot\cdot \\ 0 \end{pmatrix}$ for all p. By Lemma 9, applied to each row of $M_G^p L$, for each p there exists a matrix of homomorphisms Δ_p with $M_G^p L = \Delta_p M_H^p$. It follows that the matrix Δ_p determines a $Z_p^{\frown}$-homomorphism $\delta_p : B_p^{\frown} \to A_p^{\frown}$. Here $d'H = B \oplus \ker(g')$ has been constructed in the same manner as $d'G = A \oplus \ker(g)$. The matrix equality $M_G^p L = \Delta_p M_H^p$ can be rewritten $(v_{1p},...,v_{np})L = (\delta_p(w_{1p}), ..., \delta_p(w_{mp}))$ where v_{ip} and w_{jp} are the i-th column of M_G^p and the j-th column of M_H^p, respectively.

Let $\delta = \prod_p \delta_p : \prod_p B_p^{\frown} \to \prod_p A_p^{\frown}$. Then $(v_1,...,v_n)L = (\delta(w_1), ..., \delta(w_m))$. Since $\{v_1,...,v_n\} \subset A$ and L has integral entries, $\delta(w_i) \in A$ for $1 \le i \le m$. By definition of B, it follows that the restriction of δ to B is a homomorphism from B to A. Let $i : Q \otimes B \to Q \otimes d'H$ be the natural embedding and $\pi : Q \otimes d'G \to Q \otimes A$ be projection. Then, under our identifications, $\pi f^* i = 1 \otimes \delta$ since $(x_1^*,...,x_n^*)L = (f^*(y_1^*), ..., f^*(y_m^*))$. Thus $\pi f^* i$ is a quasi-homomorphism. But then, since A, B are reduced and $\ker(g), \ker(g')$ are divisible,

$$f^* : (Q \otimes B) \oplus \ker(g') \to (Q \otimes A) \oplus \ker(g)$$

is a quasi-homomorphism as well.

It is easy to check that d and d' satisfy $d'd \sim 1_{\mathcal{QD}}$ and $dd' \sim 1_{\mathcal{QTF}}$. This completes the proof of the theorem. ◇

One can check directly that d restricted to the full subcategory $\mathcal{QD}_0$ of torsion-free finite rank qd groups coincides with Arnold duality [A]. It is also not too difficult to check that d restricted to the embedded copy $\mathcal{QG}_t$ of $\mathcal{QG}$ in $\mathcal{QD}$ (Proposition 6) coincides with the duality between the category $\mathcal{QG}$ and the category of locally free torsion-free finite rank groups and quasi-homomorphisms, introduced by the authors in [FoW]. It follows from the construction that d, d' preserve torsion-free rank. Since, under our identifications, both df and $d'f$ are defined to be the classical vector space dual map f^* it follows that the functors d, d' are additive as well.

As an alternate method of proof of our main theorem, we could have constructed a category equivalence from $\mathcal{QD}$ to the category $\mathcal{L}$ of linear mappings to finitely presented modules over the ring of universal numbers. Then we could have applied the duality of [Fo2] from $\mathcal{L}$ to $\mathcal{QTF}$. The composite of this equivalence and duality would be the same as our duality. To make our paper reasonably self-contained, we chose to construct d directly.

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