PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 1, January 1998, Pages 239–243
S 0002-9939(98)04295-6

SPECTRAL MULTIPLICITY OF SOME STOCHASTIC PROCESSES

SLOBODANKA MITROVIC

(Communicated by James Glimm)

Abstract. In this paper we consider the connection between the canonical and the weak-canonical representations for the given second-order stochastic process in a separable Hilbert space and we extend a well-known theorem of H. Cramer concerning sufficient conditions for a process to be of multiplicity one.

Let $x(t)$, $t \in (a,b) \subset \mathbf{R}$, be a second-order real-valued process with $Ex(t) = 0$ for each t. Let $H(x,t)$ be the linear closure generated by $x(s)$, $s \in (a,t]$, in the Hilbert space H of all random variables with finite variance ($Ex^2(t) < \infty$). We will suppose that $x(t)$, $t \in (a,b)$, is continuous left and purely nondeterministic (i.e. $\bigcap_{t>a} H(x,t) = 0$). It is well known (see [1]) that there is a representation

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t,u)dz_n(u), \qquad u \le t,\ t \in (a,b), \tag{1}$$

where:

1. The processes $z_n(u)$, $n = 1, \dots, N$, are mutually orthogonal with orthogonal increments such that $Ez_n(u) = 0$ and $Ez_n^2(u) = F_n(u)$, where $F_n(u)$, $n = 1, \dots, N$, are nondecreasing functions left continuous everywhere on (a,b).
2. The nonrandom functions $g_n(t,u)$, $u \le t$, are such that:

$$Ex^2(t) = \sum_{n=1}^{N} \int_a^t g_n^2(t,u)dF_n(u) < \infty, \quad \text{for each } t \in (a,b).$$

3. $dF_1 > dF_2 > \cdots > dF_n$, where the relation $>$ means absolute continuity between measures.
4. $H(x,t) = \sum_{n=1}^{N} \bigoplus H(z_n,t)$, $t \in (a,b)$.

The expansion (1) satisfying the conditions 1, 2, 3 and 4 is the *canonical representation* or Cramer representation for the process $x(t)$. The number N (finite or infinite) is called the *multiplicity* of $x(t)$, and N is uniquely determined by the process $x(t)$. But, the processes $z_n(u)$ and the functions $g_n(t,u)$ are not uniquely determined.

Received by the editors August 24, 1995.

1991 *Mathematics Subject Classification.* Primary 60G12.

Key words and phrases. Second-order stochastic processes, canonical representation, spectral multiplicity.

This paper was presented at the 902nd AMS Meeting held at Burlington, Vermont, August 6–8, 1995.

For finite N, the representation (1) is canonical if and only if the family $\{g_n(t,u)\}_{n=1,\dots,N}$ is complete in the space $L^2(dF(u))$, $dF=\{dF_n\}_{n=1,\dots,N}$ (see Lemma 3.1 of Cramer [1]). If condition 4 in the representation (1) is replaced by the weaker condition

$$P_{H(x,s)}x(t)=\sum_{n=1}^{N}\int_a^s g_n(t,u)dz_n(u),\quad \text{for all } s\le t, s,t\in(a,b),$$

where $P_{H(x,s)}$ is the projection operator on $H(x,s)$, then (1) is said to be a *weak-canonical representation* of $x(t)$.

The kernel $\{g_n(t,u)\}_{n=1,\dots,N}$ of the weak-canonical representation need not be complete in the space $L^2(dF(u))$. Every canonical representation is the weak-canonical one (see [1], page 10). The converse need not hold. This fact is shown in the next simple example.

Example 1. If we have two mutually orthogonal stationary processes given by canonical representations:

$$x_1(t)=\int_{-\infty}^{t} g_1(t-u)dz_1(u),$$
$$x_2(t)=\int_{-\infty}^{t} g_2(t-u)dz_2(u),\qquad u\le t,\ u,t\in(-\infty,\infty),$$

then the representation of their sum, $x(t)=x_1(t)+x_2(t)$, is weak-canonical if and only if $f_1(u)=a\cdot f_2(u)$, where $f_1(u),f_2(u)$ are spectral densities, $a=$ const., but it is not canonical (see [4]).

Main result. One of the problems here is to determine the class of processes with multiplicity $N=1$. Cramer stated in Theorem 5.1 in [1] that the *regularity conditions* ensure a multiplicity of unity for a process which has a canonical expansion. Here the same result is proved for a process which has only a weak-canonical representation.

Theorem. *Let X be the class of all processes $x(t)$ admitting a weak-canonical expansion* (1)*, with N finite and (a,b) a finite subinterval of R, so that the following regularity conditions are satisfied:*

R_1. *The functions $g_n(t,u)$ and $\partial g_n(t,u)/\partial t$ are bounded and continuous for $u,t\in(a,b)$, $u\le t$.*

R_2. *$g_n(t,t)=1$, $n=1,\dots,N$, for all $t\in(a,b)$.*

R_3. *The function $F_n(u)=Ez_n^2(u)$ is absolutely continuous and not identically constant with $f_n(u)=\partial F_n(u)/\partial u$, $n=1,\dots,N$, having at most finitely many discontinuity points in any finite subinterval of (a,b).*

Then, every $x(t)\in X$ has multiplicity $N=1$.

Proof. Let us suppose that the multiplicity M of $x(t)$ is >1. For example let $M=2$. Then, there exists a canonical representation of $x(t)$ of the form:

$$x(t)=\sum_{k=1}^{2}\int_a^t G_k(t,u)dw_k(u),\qquad u\le t,t\in(a,b),\tag{2}$$

where the family $\{G_k(t,u)\}_{k=1,2}$ is complete in the space $L^2(d\Phi(u))$, $\Phi(u)=(\Phi_1(u),\Phi_2(u))$, $\Phi_k=Ew_k^2$, $k=1,2$. Without loss of generality, we can assume that the functions Φ_k are absolutely continuous. Then, from that and the condition

3 (for a canonical representation), we can find a finite subinterval $(a_1, b_1) \subset (a, b)$ where both $\Phi'_k = \varphi_k$, $k = 1, 2$, are different from zero for all $u \in (a_1, b_1)$ (see [1]).

The main idea of the proof is to show that there exists $y(t)$ from the space $H(w_1, t) \oplus H(w_2, t)$, such that $0 < Ey^2 < \infty$, and y is orthogonal to $x(s)$ for all $a < s \le t$. It means the representation (2) is not canonical and then the multiplicity is not two.

From the hypotheses of the Theorem, it follows that $x(t)$ admits a weak-canonical representation (1), which satisfies the regularity conditions. Our first step is to find a connection between a canonical and a weak-canonical representation. Both representations (1) and (2) are weak-canonical and hence for all $s < v < t$:

$$\sum_{n=1}^{N} \int_s^v g_n(t, u) dz_n(u) = \sum_{k=1}^{2} \int_s^v G_k(t, u) dw_k(u),$$

where $z_n(u)$ and $w_k(u)$, $n = 1, \ldots, N$, $k = 1, 2$, are the processes with mutually orthogonal increments on disjoint intervals. Let us construct a structural measure γ_{nk} as follows:

$$E(z_n(s) - z_n(t)) \cdot (w_k(s') - w_k(t')) = \gamma_{nk}([s, t) \cap [s', t')).$$

The finite measure with sign $d\gamma_{nk}$ is absolutely continuous with respect to the measures dF_n and $d\Phi_k$, so we may write $d\gamma_{nk}(u) = a_{nk}(u) d\Phi_k(u)$, $n = 1, \ldots, N$, $k = 1, 2$. Using the scalar product of the previous form with $w_k(u)$ we first obtain for all $s < v < t$, $k = 1, 2$:

$$\sum_{n=1}^{N} \int_s^v g_n(t, u) d\gamma_{nk}(u) = \int_s^v G_k(t, u) d\Phi_k(u),$$

and hence, $G_k(t, u) = \sum_{n=1}^{N} g_n(t, u) a_{nk}(u)$, almost everywhere with respect to the measure $d\Phi_k$, $k = 1, 2$, $u \le t$, $t \in (a, b)$. So, we may write a canonical representation (2) of $x(t)$ in the following form:

$$x(t) = \sum_{k=1}^{2} \int_a^t \left[\sum_{n=1}^{N} a_{nk}(u) g_n(t, u) \right] dw_k(u), \qquad u \le t, t \in (a, b). \tag{3}$$

Let us consider the functions $\sum_{n=1}^{N} a_{nk}(u)$, $u \in (a, b)$, $k = 1, 2$. The second step in the proof is to find a set where both $\sum a_{n1}(u)$ and $\sum a_{n2}(u)$ are different from zero. If there are no points $u \in (a_1, b_1)$ such that on the interval $(u - \delta, u)$, $\delta \ne 0$, both $\sum a_{n1}(u)$ and $\sum a_{n2}(u)$ are different from zero, then, according to assumptions about $\varphi_k \ne 0$ and conditions for $g_n(t, u)$, the process x receives the impulse $M(u)$ successively from w_1 or w_2 during the interval (a_1, b_1) (see [2]). According to [2] this means multiplicity is one: $M = \sup_{u \in (a_1, b_1)} M(u) = \sup_{u \in (a, b)} M(u) = 1$. So, let (a_2, b_2) be a finite subinterval of (a_1, b_1), such that $\sum_n a_{nk}(u) \ne 0$, for $u \in (a_2, b_2)$, $k = 1, 2$, and $0 \notin (a_2, b_2)$.

Arguing as in [1], let t be any point in (a_2, b_2) and let $h(u) = (h_1(u), h_2(u))$ be a function in $L^2(d\Phi(u))$, such that:

$$\sum_{k=1}^{2} \int_{a_2}^{s} \left[\sum_{n=1}^{N} a_{nk}(u) g_n(s, u) \right] h_k(u) \varphi_k(u) du = 0, \quad \text{for all } s \le t.$$

We will show that such $h(u) \neq 0$ exists. By conditions R_1 and R_2 this relation may be differentiated with respect to s :

$$\sum_{k=1}^{2}\left\{\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{nk}(u)\partial g_n(s,u)/\partial s\right] h_k(u)\varphi_k(u)du + \sum_{n=1}^{N} a_{nk}(s)h_k(s)\varphi_k(s)\right\} = 0.$$

This equation is satisfied if for example:

$$\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{n1}(u)\partial g_n(s,u)/\partial s\right] h_1(u)\varphi_1(u)du + \sum_{n=1}^{N} a_{n1}(s)h_1(s)\varphi_1(s) = 1 \tag{4}$$

and

$$\int_{a_2}^{s}\left[\sum_{n=1}^{N} a_{n2}(u)\partial g_n(s,u)/\partial s\right] h_2(u)\varphi_2(u)du + \sum_{n=1}^{N} a_{n_2}(s)h_2(s)\varphi_2(s) = -1. \tag{5}$$

These are the nonhomogeneous Volterra integral equations of the second kind with unknown functions $h_k(s)\varphi_k(s)$, $s \in (a_2, t]$, $k = 1, 2$. Let us consider the first of them. By the restriction imposed on $\partial g_n(s,u)/\partial s$, $n = 1, \dots, N$, there exists a solution $h_1(s)\varphi_1(s)$, $s \in (a_2, b_2)$, not equal to zero almost everywhere if the following conditions hold:

$$\int_{a_2}^{b_2}\left[\sum_{n=1}^{N} a_{n1}(u)\right]^2 du < \infty \quad \text{and} \quad \int_{a_2}^{b_2}\left[\sum_{n=1}^{N} a_{n1}(s)\right]^{-2} ds < \infty$$

(see [3]).

By condition R_1 for $g_n(t,u)$ and the fact that $G_1(t,u) \in L^2(\varphi_1(u)du)$, it is easy to see that $\sum a_{n1}(u) \in L^2(\varphi_1(u)du)$. As $\varphi_1 > 0$ and $\sum a_{n1} \neq 0$ on (a_2, b_2), it follows that $(\sum a_{n1})^2 \le (\sum a_{n1})^2\varphi_1$, for $\varphi_1 \ge 1$, or $\varepsilon \cdot (\sum a_{n1})^2 \le (\sum a_{n1})^2\varphi_1$, for $0 < \varepsilon \le \varphi_1 \le 1$. Hence, it is clear that $\sum a_{n1}(u) \in L^2(du)$, and that $[\sum a_{n1}(u)]^{-1} \in L^2(du)$, on the finite subinterval (a_2, b_2), which does not contain 0. So, a solution $h_1(s)\varphi_1(s)$, $s \in (a_2, b_2)$, of the integral equation (4) exists.

The same holds for the integral equation (5). Since $\varphi_k \neq 0$, $k = 1, 2$, on (a_2, b_2), it follows that:

$$\int_{a_2}^{t} h_1^2(u)d\Phi_1(u) + \int_{a_2}^{t} h_2^2(u)d\Phi_2(u) > 0, \quad \text{for all } t \in (a_2, b_2).$$

This means that the family $\{G_k(t,u)\}_{k=1,2}$ is not complete in the space $L^2(d\Phi(u))$, and multiplicity of $x(t)$ is not 2. Using similar arguments we see that the multiplicity cannot be any natural number > 1. The proof is completed. $\square$

Note. The statement of the Theorem is valid even if we assume that (a, b) is an infinite subinterval of R.

Example 2. Let $x(t) = \int_{-\infty}^{t} e^{-c(t-u)}dz_1(u) + \int_{-\infty}^{t} d \cdot e^{-c(t-u)}dz_2(u)$, $u \le t, u, t \in R$, be a process, where $z_1(u)$ and $z_2(u)$ are the mutually orthogonal processes with orthogonal increments such that $Ez_n(u) = 0$, $Ez_n^2(u) = f_n(u)du$, $n = 1, 2, d =$ const ., $f_1(u) = 2c$, $f_2(u) = 2cd^2$. Clearly, $x(t)$ has a weak-canonical representation and since it satisfies the regularity conditions, it has multiplicity one.

Example 3. Let $x(t)$, $0 \le t \le \tau$, be represented by $x(t) = c_1z_1(t) + c_2z_2(t) + \cdots + c_Nz_N(t)$, where $z_n(t)$ are independent Wiener processes, $c_n =$ const., $n = 1, \dots, N$. This representation is weak-canonical because $x(t) - x(s)$ is orthogonal to $x(s)$ for all $s < t$, and $P_{H(x,s)}x(t) = x(s)$. Since the regularity conditions are satisfied, $x(t)$ has multiplicity one.

References

[1] H. Cramer, *Structural and Statistical Problems for a Class of Stochastic Processes*, Princeton University Press, Princeton, New Jersey, 1971, pp. 30. MR **53:**4204

[2] ———, *Stochastic Processes as Curves in Hilbert Space*, Theory Probab. Appl., Tom. **9** (1964), 193–204.

[3] Frédéric Riesz and Béla Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Akadémiai Kiadó, Budapest, 1972 (French); translated by the Amer. Math. Soc., 1974.

[4] T. N. Siraja, *Canonical representations of second order random processes*, Teor. Veroyatnost. i Primenen. **12** (1977), 429–435. (Russian) MR **56:**9664

[5] S. Mitrovic, *A note concerning a theorem of Cramer*, Proceedings of the Amer. Math. Soc., **121** (2) (1994), 589–591. MR **94h:**60050

Ljutice Bogdana 2/2 No. 35, Belgrade 11040, Serbia
E-mail address: emitrosl@ubbg.etf.bg.ac.yu