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MICROLOCAL ANALYSIS OF ULTRADISTRIBUTIONS

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Abstract. The ultradistributional wave front sets of an ultradistribution u are characterized by the behaviour of $K*u$ on the boundary of the tube domain $D\mathbf{R}^n$, where K is the kernel analysed by Hörmander.

1. Introduction

We follow the results and ideas of Chapter 8 in [2] and characterize the ultradistributional wave front sets of type WF^* and WF_* for a tempered ultradistribution u by using its convolution with the kernel K (cf. [2]). We give the assertions which are analogous to the corresponding ones for distributions.

The paper is organized as follows. In Section 2 we recall the definitions of ultradistributional wave front sets WF^* and WF_*. These notions are introduced by Eida [1] and Komatsu [4] and called singular spectrums SS^* and SS_*. In Section 3 we recall the definition of tempered ultradistribution spaces $\mathcal{S}'^*$ (cf. [5], [7]) and a theorem from [8] which is needed for later use. In Section 4 we give the microlocal analysis of a $u \in \mathcal{S}'^*$ (Theorems 2, 3 and Corollary 1). Necessary and sufficient conditions for an analytic function in the tube domain $D\mathbf{R}^n$ which determine its boundary value in $\mathcal{S}'^*$ are given in Theorem 2.

2. Notation and notions

By $M_p,\ p \in \mathbf{N}_0$, we denote a sequence of positive numbers with $M_0 = 1$. We refer to [3] and [6] for the meaning of conditions (M.1), (M.2)$'$, (M.2), (M.3)$'$ and (M.3). We also use the following condition ([6]):

(M.1)* $\quad M^*_{p-1}M^*_{p+1} \le M^{*}_p, p \in \mathbf{N}$, where $M^*_0 = 1, M^*_p = M_p/p!,\ p \in \mathbf{N}$.

Let M_p satisfy (M.1) and (M.3)$'$. The associated function $M(\rho)$ and the growth function $\tilde{M}(\rho)$ related to M_p are defined by

$$M(\rho) = \sup_{p\in\mathbf{N}_0} \ln \frac{\rho^p}{M_p},\ \ \tilde{M}(\rho) = \sup_{p\in\mathbf{N}_0} \ln \frac{\rho^p}{M^*_p},\ \rho > 0.$$

Note, for a given $L > 0$ there is $L_1 > 0$ such that ([8])

$$M(L|\xi|) - |\eta||\xi| \le \tilde{M}(L_1/|\eta|),\ \ \xi, \eta \in \mathbf{R}^n. \tag{1}$$

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We denote by Ω an open set in $\mathbf{R}^n$, and $K \subset\subset \Omega$ means that K is a compact subset of Ω. Recall,

$$\|\varphi\|_{K,h,M_p} = \sup_{x\in K,\alpha\in\mathbf{N}_0^n} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}}, \varphi \in C^\infty(\Omega).$$

The symbol $*$ is used for both (M_p) and $\{M_p\}$. We refer to [3] for the definitions of $\mathcal{D}'^*(\Omega), \mathcal{D}^*_K(\Omega)$ and $\mathcal{D}'^*_K(\Omega)$. Throughout the paper we will assume that (M.1), (M.2)$'$ and (M.3)$'$ hold. Eida [1] and Komatsu [4] have defined SS_*- and SS^*-singular support of a hyperfunction f. We will recall the definitions related to ultradistributions and call them wave front sets.

Let N_p be a sequence of positive numbers which satisfies (M.1), (M.2)$'$, (M.3)$'$ and $N_0 = 1$. Then ([3]):

$$(N_p) \le (M_p) \quad (\text{resp., } \{N_p\} \le \{M_p\})$$

if there are constants $L > 0$ and $C > 0$ (resp., for every $\epsilon > 0$ there is $C_\epsilon > 0$) such that

$$N_p \le CL^pM_p \quad (\text{resp., } \; N_p \le C_\epsilon \epsilon^p M_p), \;\; p \in \mathbf{N}_0.$$

Let $f \in \mathcal{D}'^\dagger$, where $\dagger = N_p \le * = M_p$. Then $(x,\omega) \in S^*\Omega = \Omega \times S^{n-1}$ is not in WF_*f (resp., not in WF^*f) iff there exist a neighborhood $U \subset \Omega$ of x and a conic neighborhood Γ of ω of the form $\Gamma = \{\xi \ne 0;\; |\xi/|\xi| - \omega| < \eta\}$ such that for every $\phi \in \mathcal{D}^*(U)$ the following hold:

In the (M_p) case, for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \le C_\epsilon e^{-M(\epsilon|\xi|)}$, $\xi \in \Gamma$ (resp., there are $k > 0$ and $C > 0$ such that $|\widehat{\phi f}(\xi)| \le Ce^{M(k|\xi|)}$, $\xi \in \Gamma$).

In the $\{M_p\}$ case, there exist $k > 0$ and $C > 0$ such that $|\widehat{\phi f}(\xi)| \le Ce^{-M(k|\xi|)}$, $\xi \in \Gamma$ (resp., for every $\epsilon > 0$, there is $C_\epsilon > 0$ such that $|\widehat{\phi f}(\xi)| \le C_\epsilon e^{M(\epsilon|\xi|)}$, $\xi \in \Gamma$). Note, the notion $WF_{\{M_p\}}$ is equal to Hörmander's notion WF_L.

The definition of the singular spectrum SSf, where $f \in \mathcal{B}(\Omega)$, is given by Sato (cf. [10]). For an $f \in \mathcal{D}'^*(\Omega)$, $(x,\omega) \in S^*\Omega$ is not in SSf if this point is not in $SS\{f\}$, where $\{f\}$ denotes the corresponding hyperfunction. This notion is equal to Hörmander's WF_Af, the analytic wave front set of f ([2], Definition 9.3.2, Theorem 9.6.3) and we will use this notation.

We also use the definition according to which $(x,\xi) \in \Omega \times (\mathbf{R}^n \setminus \{0\})$ is an element of the corresponding singular spectrum defined above if this holds for $(x, \xi/|\xi|)$.

Let $D\mathbf{R}^n = \{z \in \mathbf{C}^n; |\operatorname{Im} z| < 1\}$ and $S^*\mathbf{R}^n = \partial D\mathbf{R}^n$. Recall ([4]), $O^*|_{D\mathbf{R}^n}$ (resp., $O_*|_{D\mathbf{R}^n}$) is a sheaf over $\mathbf{C}^n$ of holomorphic functions in $D\mathbf{R}^n$ which satisfy the following growth condition near $S^*\mathbf{R}^n$:

Let U be an open set in $\mathbf{C}^n$. Then a function $F(z)$ is in $O^*|_{D\mathbf{R}^n}(U)$ (resp., $O_*|_{D\mathbf{R}^n}(U)$) if F is holomorphic in $D\mathbf{R}^n \cap U$ such that for every compact set $K \subset\subset U$,

in the (M_p) case, there are $C > 0$ and $k > 0$,

in the $\{M_p\}$ case, for every $k > 0$ there is $C > 0$ such that

$$|F(z)| \le Ce^{\tilde{M}(\frac{k}{1-|y|})}, \;\; z = x + \sqrt{-1}y \in K \cap D\mathbf{R}^n$$

(resp., for every compact set $K \subset\subset U$,

in the (M_p) case, for every ultradifferential operator $P(\partial)$ of class (M_p),

in the $\{M_p\}$ case, for every ultradifferential operator $P(\partial)$ of class $\{M_p\}$

$$P(\partial_z)F \quad \text{is bounded in} \quad K \cap D\mathbf{R}^n.)$$

3. Tempered ultradistributions

We will recall the definitions and the basic structural properties of tempered ultradistributions (cf. [5] and [7]). Let $m > 0$. The space of smooth functions φ on $\mathbf{R}^n$ which satisfy

$$\sigma_{m,2}(\varphi) = \Big(\sum_{\alpha,\beta\in\mathbf{N}_0^n}\int_{\mathbf{R}^n} \mid \frac{m^{|\alpha+\beta|}}{M_{|\alpha|}M_{|\beta|}}(1+\mid x\mid^2)^{|\beta|/2}\varphi^{(\alpha)}(x)\mid^2 dx\Big)^{1/2} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,2}$, is denoted by $\mathcal{S}_2^{M_p,m}$. The strong duals of $\mathcal{S}^{(M_p)} = \text{ proj }\lim_{m\to\infty}\mathcal{S}_2^{M_p,m}$ and $\mathcal{S}^{\{M_p\}} = \text{ ind }\lim_{m\to 0}\mathcal{S}_2^{M_p,m}$ are called spaces of tempered ultradistributions of Beurling and Roumieau type. For every fixed $p \in [1,\infty]$, the family of norms $\{\sigma_{m,2};\ m > 0\}$ is equivalent to the family of norms $\{\sigma_{m,p};\ m > 0\}$ where instead of the L^2 norm we put the L^p norm. In fact, in the sequel we will use the family of norms

$$s_h(\phi) = \sup\{\frac{h^{|\alpha|}}{M_{|\alpha|}} \mid \phi^{(\alpha)}(x) \mid e^{M(h|x|)}; \alpha\in\mathbf{N}_0^n, x\in\mathbf{R}^n\},\ \ h > 0,$$

which is equivalent to $\{\sigma_{h,2};\ h > 0\}$.

$\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are (FS)- and (LS)-spaces respectively. If (M.2) holds, they are (FN)- and (LN)-spaces respectively and $\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*$, $\mathcal{S}^* \hookrightarrow \mathcal{S}$, where "$A \hookrightarrow B$" means that A is dense in B and the inclusion mapping is continuous.

An $f \in \mathcal{D}'^*$ is in $\mathcal{S}'^*$ if and only if there exists a family $F_{\alpha,\beta}$, $\alpha,\beta \in \mathbf{N}_0^n$, in $L^2(\mathbf{R}^n)$ such that

$$f = \sum_{\alpha,\beta\in\mathbf{N}_0^n}((1+\mid x\mid^2)^{\beta/2}F_{\alpha,\beta})^{(\alpha)} \quad \text{in} \quad \mathcal{S}'^*,$$

and in the case $\mathcal{S}'^{(M_p)}$, there exists $k > 0$ (resp., in the case $\mathcal{S}'^{\{M_p\}}$, for every $k > 0$) such that

$$\Big(\sum_{\alpha,\beta\in\mathbf{N}_0^n}\int_{\mathbf{R}^n} \mid \frac{M_{|\alpha|}M_{|\beta|}}{k^{|\alpha+\beta|}}F_{\alpha,\beta}(x)\mid^2\Big)^{1/2} < \infty.$$

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto itself.

We recall the next theorem for later use.

Theorem 1 ([8]). *Assume* (M.1)* , (M.2) *and* (M.3)$'$ *hold for* $\dagger = N_p$ $(N_0 = 1)$. *Let* Γ *be an open convex cone in* $\mathbf{R}^n$ *and* F *be an analytic function in*

$$Z = \{z\in\mathbf{C}^n;\ \operatorname{Im} z\in\Gamma,\ |\operatorname{Im} z| < \gamma\}$$

for some $\gamma > 0$. *Moreover, assume*

$$|F(x+\sqrt{-1}y)| \le C_{a,b}e^{N(a|x|)+\tilde{N}(\frac{b}{|y|})},\ \ x+\sqrt{-1}y\in Z,$$

in the (N_p) *case for some* $a > 0$, $b > 0$, *and* $C_{a,b} > 0$, *and in the* $\{N_p\}$ *case for every* $a > 0$, $b > 0$ *there exists* $C_{a,b} > 0$. *Then*

$$F(x+\sqrt{-1}y) \overset{\mathcal{S}'^{\dagger}}{\to} F(x+\sqrt{-1}0),\ \ y\to 0,\ \ y\in\Gamma.$$

Note ([8]),

$$\langle F(x+\sqrt{-1}0),\varphi(x)\rangle = \int_{\mathbf{R}^n} F(x+\sqrt{-1}Y)\Phi(x+\sqrt{-1}Y)dx \tag{2}$$

$$+2\sqrt{-1}\sum_{i=1}^{n} Y_i \int_0^1 \int_{\mathbf{R}^n} \frac{\partial}{\partial \bar{z}_i}\Phi(x+\sqrt{-1}tY)F(x+\sqrt{-1}Yt)dtdx.$$

Moreover, there exists $C>0$ such that

$$|\langle F(x+\sqrt{-1}0),\varphi(x)\rangle| \leq C s_h(\varphi),\ \varphi\in\mathcal{S}^{\dagger}, \tag{3}$$

where $\Phi(z)$ is the almost analytic extension of φ (cf. [6] or [8]).

4. Microlocal analysis of ultradistributions

As in [2], put

$$I(\xi)=\int_{|\omega|=1} e^{-\langle\omega,\xi\rangle}d\omega, \xi\in\mathbf{R}^n,\ K(z)=(2\pi)^{-n}\int \frac{e^{\sqrt{-1}\langle z,\xi\rangle}}{I(\xi)}d\xi, z\in D\mathbf{R}^n.$$

We call K Hörmander's kernel. Recall ([2], Lemma 8.4.10) that $K(z)$ is an analytic function in $\tilde{\Omega}=\{z\in\mathbf{C}^n;\ \langle z,z\rangle\notin(-\infty,-1]\}$. The properties of this function given in the quoted lemma imply

$$|K(x+\sqrt{-1}y)|\leq \frac{Cn!e^{-c|x|}}{(1-|y|)^n},\ \ z\in D\mathbf{R}^n,$$

for some $C>0$ and $c>0$. By using the Cauchy formula on the contour

$$\Gamma_{t+\sqrt{-1}y}=\Gamma_{t_1+\sqrt{-1}y_1}\times\cdots\times\Gamma_{t_n+\sqrt{-1}y_n},$$

where $\Gamma_{t_j+\sqrt{-1}y_j}$ is the boundary of

$$\{(\tau_j,u_j);\ |t_j-\tau_j|<1,\ |y_j-u_j|<(1-|y|)/2\},$$

it follows that for some $C>0$ and $c>0$

$$|\frac{\partial^\alpha}{\partial t^\alpha}K(t+\sqrt{-1}y)|<C\frac{\alpha!n!}{(1-|y|)^{|\alpha|+n}}e^{-c|t|},\ t\in\mathbf{R}^n,\ |y|<1,\ \ \alpha\in\mathbf{N}_0^n.$$

This implies that $K(\cdot+\sqrt{-1}y)\in\mathcal{S}^*$ for every fixed y, $|y|<1$.

The next theorem is proved in [2] for tempered distributions.

Theorem 2. *Let* $\dagger=N_p$ *and* $*=M_p$ *satisfy* (M.1)* , (M.2) *and* (M.3)$'$ *and let* $N_p\leq M_p$. *Let* $u\in\mathcal{S}'^{\dagger}$ *and* $U(z)=(u*K)(z)=\langle u(t),K(x-t+\sqrt{-1}y)\rangle,\ z\in D\mathbf{R}^n$.

a) *Then* U *is analytic in* $D\mathbf{R}^n$ *and*

in the (N_p) *case, for some* $a>0,\ b>0$ *there is* $C_{a,b}>0$,

in the $\{N_p\}$ *case, for every* $a>0,\ b>0$ *there is* $C_{a,b}>0$, *such that*

$$|U(z)|\leq C_{a,b}e^{N(a|x|)+\tilde{N}(b/(1-|y|))},\ z=x+\sqrt{-1}y\in D\mathbf{R}^n. \tag{4}$$

For every $\omega\in S^{n-1}$ *there exists the limit*

$$\lim_{\substack{y\to\omega\\ z\in D\mathbf{R}^n}}\langle U(x+\sqrt{-1}y),\phi(x)\rangle=\langle U(x+\sqrt{-1}\omega),\phi(x)\rangle,\ \phi\in\mathcal{S}^{\dagger},$$

and the mapping $S^{n-1}\ni\omega\mapsto U(\cdot+\sqrt{-1}\omega)\in\mathcal{S}'^{\dagger}$ *is continuous. Moreover,*

$$\langle u,\phi\rangle=\int_{\mathbf{R}^n}\langle U(\cdot+\sqrt{-1}\omega),\phi\rangle d\omega,\ \ \phi\in\mathcal{S}^{\dagger}. \tag{5}$$

b) *Conversely, if* U *is given satisfying* (4)*, then* (5) *defines an ultradistribution in* $\mathcal{S}'^{\dagger}$.

c1) $q = (x_0, \omega_0) \notin WF^*u$ *if and only if* U *is* $\mathcal{O}^*$ *in a neighborhood of* $x_0 - \sqrt{-1}\omega_0$.

c2) $q \notin WF_*u$ *if and only if* U *is* $\mathcal{O}_*$ *in a neighborhood of* $x_0 - \sqrt{-1}\omega_0$.

c3) $q \notin WF_Au$ *if and only if* U *is analytic at* $x_0 - \sqrt{-1}\omega_0$ *(i.e. in a neighborhood of this point).*

Proof. a) We will give only the proof in the (M_p) case. There exist $h > 0$, $C_h > 0$ and suitable constants which do not depend on h, such that

$$
\begin{aligned}
|\langle u(t), K(z-t)\rangle| &\le C_h \sup_{\substack{\alpha\in\mathbf{N}_0^n\\ t\in\mathbf{R}^n}} \{\frac{h^\alpha|\frac{\partial^\alpha}{\partial t^\alpha}K(z-t)|e^{N(h|t|)}}{N_{|\alpha|}}\} \\
&\le CC_h \sup_{\substack{\alpha\in\mathbf{N}_0^n\\ t\in\mathbf{R}^n}} \{\frac{h^\alpha \alpha! n! e^{-c|x-t|}}{N_{|\alpha|}(1-\mid y\mid)^{|\alpha|+n}} e^{N(h|t|)}\} \\
&\le C_1 C_h e^{N(h|x|)} \sup_{\alpha\in\mathbf{N}_0^n} \frac{h^\alpha \alpha!}{(1-\mid y\mid)^{|\alpha|}N_{|\alpha|}} \\
&\le C_1 C_h e^{N(h|x|)+\tilde{N}(\frac{h}{1-|y|})},\ z \in D\mathbf{R}^n.
\end{aligned}
$$

This proves (4). Since the Fourier transformation of $U(\cdot + \sqrt{-1}y)$, $|y| < 1$, is $e^{-\langle y,\xi\rangle}\hat{u}(\xi)/I(\xi), \xi \in \mathbf{R}^n$, and

$$
I(\xi) = (2\pi)^{(n-1)/2} \frac{e^{|\xi|}}{\mid \xi\mid^{(n-1)/2}}(1 + \mathcal{O}(1/|\xi|)),\ |\xi| \to \infty \tag{6}
$$

([2], p. 287), it follows that $y \mapsto U(\cdot + \sqrt{-1}y)$, $|y| \le 1$, is a continuous function $\{y; |y| \le 1\} \to \mathcal{S}'^{(N_p)}$.

Let us prove this. For any $\phi \in \mathcal{S}^\dagger, \xi \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_0$ and $|y| \le 1$ we have

$$
\begin{aligned}
&\mid (e^{-\langle y,\xi\rangle}(1/I(\xi))\phi(\xi))^{(\alpha)} \mid \\
&=\mid \sum_{i\le\alpha}\binom{\alpha}{i}(y)^{\alpha-i}e^{-\langle y,\xi\rangle}\sum_{p\le i}\binom{i}{p}\phi^{(i-p)}(\xi)(1/I(\xi))^{(p)} \mid .
\end{aligned}
$$

Since $|(I(\xi))^{(\beta)}| \le I(\xi)$, $\xi \in \mathbf{R}^n$, $\beta \in \mathbf{N}_0^n$, we conclude

$$
|(1/I(\xi))^{(p)}| \le 2^{|p|}p!/I(\xi),\ \ \xi \in \mathbf{R}^n.
$$

Now by (6),

$$
|(e^{-\langle y,\xi\rangle}(1/I(\xi))\phi(\xi))^{(\alpha)}| \le C3^{|\alpha|}(1+|\xi|)^{(n-1)/2}\sup_{|i|\le\alpha}|\phi^{(i)}(\xi)|
$$

which implies:

$$
\begin{aligned}
\langle u, \phi\rangle &= \lim_{r\to 1}\left\langle \int_{|\omega|=1}\frac{e^{\langle -r\omega,\xi\rangle}\hat{u}(\xi)}{I(\xi)}d\omega, \hat{\phi}(\xi)\right\rangle \\
&= \lim_{r\to 1}\int_{|\omega|=1}\left\langle \frac{e^{-r\langle\omega,\xi\rangle}\hat{u}(\xi)}{I(\xi)},\ \hat{\phi}(\xi)\right\rangle d\omega = \int_{|\omega|=1}\langle U(x+\sqrt{-1}\omega), \phi(x)\rangle d\omega,
\end{aligned}
$$

where $y = r\omega$, $\omega \in S^{n-1}$.

b) Conversely, if U is analytic in $D\mathbf{R}^n$ and satisfies (4), then Theorem 1, more precisely (2) and (3), imply that

$$\lim_{\substack{y\to\omega \\ z\in D\mathbf{R}^n}} U_y = U_\omega, \quad |(\omega_1,\dots,\omega_n)| = 1,$$

exists in $\mathcal{S}'^{(N_p)}$, where $U_y = U(\cdot + \sqrt{-1}y),\ |y| < 1$ and

$$\langle U_\omega, \phi\rangle = \int U(x)\Phi(x-\sqrt{-1}\omega)dx$$

$$-2\sqrt{-1}\sum \omega_i \int_0^1 \int_{\mathbf{R}^n} U(x+\sqrt{-1}(1-t)\omega)\frac{\partial\Phi}{\partial\bar{z}_1}(x+\sqrt{-1}(1-t)\omega)dtdx.$$

In fact, we have to put $U(z+\sqrt{-1}\omega) = F(z),\ Y = -\omega$, and the conclusion follows as in Theorem 1.

Moreover, it follows that $S^{n-1} \ni \omega \mapsto U_\omega \in \mathcal{S}'^{(N_p)}$ is continuous, which implies that by

$$\phi \mapsto \langle u, \phi\rangle = \int \langle U_\omega(x), \phi(x)\rangle d\omega,\ \phi \in \mathcal{S}'^{(M_p)},$$

an element from $\mathcal{S}'^{(N_p)}$ is defined. One can easily show that $U = u * K$.

c) We will prove the "if" parts.

Suppose $q \notin WF^*u$. Let $\varphi \in \mathcal{D}^*(U)$, where $U = \{x; |x-x_0| < 2r\}$ and $0 \le \varphi \le 1,\ \varphi \equiv 1$ on $\{x; |x-x_0| < r\}$.

Note that

$$|\widehat{\varphi u}(\xi)| < C_1 e^{M(L_1|\xi|)},\ \ \xi \in \mathbf{R}^n, \tag{7}$$

for some $L_1 > 0$ and some $C_1 > 0$ in the (M_p) case (resp., for every $L_1 > 0$ there is $C_1 > 0$ in the $\{M_p\}$ case). Put $U = \varphi u * K + (1-\varphi)u * K = U_1 + U_2$. This decomposition is also used in parts b) and c). Let $x \in \{x; |x-x_0| < r/2\}$. Then,

$$(1-\varphi(t))K(x+\sqrt{-1}y-t) = 0,\ \ |t-x_0| < r,\ \ x+\sqrt{-1}y \in \mathbf{C}^n.$$

If $|t-x_0| \ge r$, then $|x-t| \ge r/2$ and for $|y| < 1 + r/2$,

$$t \mapsto K(x+\sqrt{-1}y-t)$$

is a function which belongs to S^*.

This implies that

$$U_2(z) = \langle u(t), (1-\varphi(t)), K(z-t)\rangle,\ \ z \in D\mathbf{R}^n,$$

is analytic in

$$\{x+\sqrt{-1}y;\ |y| < 1 + r/2,\ |x-x_0| < r/2\},$$

which is a neighborhood of $x_0 - \sqrt{-1}\omega_0$. So in the proofs of c1), c2) and c3) below we have to consider U_1.

c1) Let γ be an open convex cone which contains ω_0, $\epsilon > 0$ and $|y+\omega_0| < \epsilon$, $|y| < 1$. We shall use the following inequalities:

$$M(L|\xi|) - \langle \xi, y\rangle - |\xi| \le \tilde{M}(\tilde{L}/(1-|y|)),\ \ \xi \in \gamma, \tag{8}$$

which holds for some $\tilde{L} > 0$ (see (1)), and

$$-\langle y, \xi\rangle - |\xi| \le -\epsilon|\xi|,\ \ \xi \notin \gamma, \tag{9}$$

which holds if ϵ is small enough.

Let $|y+\omega_0|<\epsilon,\ |y|<1,\ |x-x_0|<r$. Then $(z=x+\sqrt{-1}y)$,

$$|U_1(z)| = |\mathcal{F}(\varphi u * K(z,\cdot))(\xi)|$$
$$\le \int_{\mathbf{R}^n} \frac{e^{-\langle y,\xi\rangle}}{I(\xi)}\widehat{\varphi u}(\xi)d\xi = \int_{\xi\in\gamma} + \int_{\xi\notin\gamma}.$$

By using (7) and (8) for the first integral and (7) and (9) for the second integral, we obtain that U_1 is $\mathcal{O}^*$ in some neighborhood of $x_0-\sqrt{-1}\omega_0$.

c2) Let $q\notin WF_*u$. By using

$$|P(\partial_z)(u\varphi * K)(z)| \le \int_{\mathbf{R}^n} \frac{e^{-\langle y,\xi\rangle}}{I(\xi)} P(\sqrt{-1}\xi)\widehat{u\varphi}(\xi)d\xi,$$

in a similar way we obtain that for every ultradifferential operator $P(\partial_z)$ of the corresponding class, $P(\partial_z)U$ is bounded in a neighborhood of $x_0-\sqrt{-1}\omega_0$.

c3) If $(x_0,\omega_0)\notin WF_Au$, then $(x_0,\omega_0)\notin WF_Au\phi$. By [2], Definition 9.3.2, it is equivalent to the analyticity of $u\varphi * K$ at $x_0-\sqrt{-1}\omega_0$.

The converse parts in c1), c2) and c3) follow from the next lemma (see [2], Lemma 8.4, for tempered distributions).

Lemma 1. *Let $d\mu$ be a measure on S^{n-1} and Γ an open convex cone in $\mathbf{R}^n$ such that*

$$\langle y,\omega\rangle < 0 \quad \textit{if} \quad 0\ne y\in\overline{\Gamma},\ \ \omega\in\operatorname{supp} d\mu.$$

Let U be analytic in $D\mathbf{R}^n$, and satisfy (4) in $D\mathbf{R}^n$. Then

$$F(z) = \int_{S^{n-1}} U(z+\sqrt{-1}\omega)d\mu(\omega),\ \operatorname{Im} z\in\Gamma, |\operatorname{Im} z|<\gamma,$$

is analytic and

$$(10) \qquad | F(z) | \le C_{a,b} e^{N(a|x|)+\tilde{N}(b/(1-|y|))},\ \operatorname{Im} z\in\Gamma,\ |\operatorname{Im} z|<\gamma,$$

for some a,b and some $C_{a,b}>0$ in the (N_p) case, and for every a,b there is $C_{a,b}>0$ in the $\{N_p\}$ case.

For every measure $d\mu$ on S^{n-1} and

$$U_\mu = \int_{S^{n-1}} U(\cdot+\sqrt{-1}\omega)d\mu(\omega),$$

there holds:

$WF^*U_\mu \subset \{(x,\omega); -\omega\in\operatorname{supp} d\mu$ and U is not $\mathcal{O}^*$ at $x-\sqrt{-1}\omega\}$.

$WF_*U_\mu \subset \{(x,\omega); -\omega\in\operatorname{supp} d\mu$ and U is not $\mathcal{O}_*$ at $x-\sqrt{-1}\omega\}$.

$WF_AU_\mu \subset \{(x,\omega); -\omega\in\operatorname{supp} d\mu$ and U is not analytic at $x-\sqrt{-1}\omega\}$.

Proof. Let $\omega\in\operatorname{supp} d\mu$ and $\operatorname{Im} z\in\Gamma$. Then for some $C>0$,

$$1-|\operatorname{Im}(z+\sqrt{-1}\omega)| > \frac{|\operatorname{Im} z|}{2} \quad \text{if } |\operatorname{Im} z|<C \quad ([5], \text{Lemma 8.4.12}).$$

This implies that we may use Theorem 1 for F. Thus, $F(\dot{+}\sqrt{-1}\Gamma 0)\in\mathcal{S}'^{\dagger}$.

Note

$$WF^*U_\mu \subset WF_*U_\mu \subset WF_AU_\mu \subset \mathbf{R}^n\times\Gamma^0,$$

where $\Gamma^0=\{\xi;\langle x,\xi\rangle\ge 0$ for every $x\in\Gamma\}$ is the dual cone.

We shall prove only the estimate for WF^*U_μ since the other parts can be proved similarly. Denote $\operatorname{sing\,supp}^* u=\pi_1 WF^*u$. There holds

$$\operatorname{sing\,supp}^* U_\mu \subset \{x; U \text{ is not } \mathcal{O}^* \text{ at } x+\sqrt{-1}\omega \text{ for some } \omega\in\operatorname{supp} d\mu\}.$$

Decompose

$$d\mu = \sum_{j=1}^{r} d\mu_j, \quad \operatorname{supp} d\mu_j \subset \operatorname{supp} d\mu \cap \Gamma_j,$$

where Γ_j are open convex cones such that $\operatorname{int} \Gamma_j^0 \neq \emptyset$. By replacing $d\mu$ and Γ from the first part of the lemma by $d\mu_j$ and $-\operatorname{int}(\Gamma_j^0)$ we obtain

$$\begin{aligned} WF^* U_\mu \quad & \subset \textstyle\bigcup_{j=1}^{r} \{(x,\xi);\ -\xi \in \overline{\Gamma_j},\ U \text{ is not } \mathcal{O}^* \\ & \text{at } x - \sqrt{-1}\omega \text{ for some } \omega \in \Gamma_j \cap S^{n-1}\}. \end{aligned}$$

If $-\frac{\xi}{|\xi|} \notin \operatorname{supp} d\mu$ or U is not $\mathcal{O}^*$ at $x - \sqrt{-1}\frac{\xi}{|\xi|}$, there exists the decomposition of $d\mu$ such that $-\xi \notin \overline{\Gamma_j}$, for $j = 1, \dots, r$ or $-\xi \notin \overline{\Gamma_j}$ for $j = 2, \dots, r$ and U is $\mathcal{O}^*$ at $x + \sqrt{-1}\omega$ for every $\omega \in \Gamma_1 \cap S^{n-1}$. In both cases $(x,\xi) \notin WF^* U_\mu$.

Immediate consequences of this theorem are the next corollary and theorem (see [2], Corollary 8.4.13 and Theorem 8.4.15 for tempered distributions).

Corollary 1. *Let G_j be closed subsets of S^{n-1} such that $\bigcup_{j=1}^{r} G_j = S^{n-1}$. Any $u \in \mathcal{S}'^{\dagger}(\mathbf{R}^n)$ can be written $u = \sum_{j=1}^{r} u_j$, $u_j \in \mathcal{S}'^{\dagger}(\mathbf{R}^n)$ $(\dagger \leq *)$ with*

a) $WF^* u_j \subset WF^* u \cap \mathbf{R}^n \times G_j,\ j = 1, \dots, r.$

b) $WF_* u_j \subset WF_* u \cap \mathbf{R}^n \times G_j,\ j = 1, \dots, r.$

c) $WF_A u_j \subset WF_A u \cap \mathbf{R}^n \times G_j,\ j = 1, \dots, r.$

If $u \in \mathcal{E}'^{\dagger}$, then u_j, $j = 1, \dots, r$, have compact supports as well.

If $u = \sum u_j'$ is another such decomposition, then

$$u_j' = u_j + \sum_k u_{jk}, \quad u_{jk} \in S'^*,$$

$$u_{jk} = -u_{jk} \quad \text{and } WF^* u_{jk} \subset (WF^* u) \cap \mathbf{R}^n \times (G_j \cap G_k).$$

The same holds for $WF_* u_{jk}$ and $WF_A u_{jk}$.

Theorem 3. *Let Γ be an open convex cone in $\mathbf{R}^n$, $u \in \mathcal{D}'^*(\Omega)$, $\Omega \subset \mathbf{R}^n$, and $WF_* u \subset \Omega \times \Gamma^0$. If $\Omega_1 \subset\subset \Omega$ and Γ_1 is an open convex cone with closure $\subset \Gamma \cup \{0\}$, then there is an F analytic in $\{x + \sqrt{-1}y;\ x \in \Omega_1,\ y \in \Gamma_1,\ |y| < \gamma\}$ such that for some $k > 0$ and $C > 0$ in the (M_p) case (resp., for every $k > 0$ there is $C > 0$ in the $\{M_p\}$ case),*

$$|F(x + \sqrt{-1}y)| \leq C e^{\tilde{M}(k/|y|)}, \ x \in \Omega_1,\ y \in \Gamma_1,\ |y| < \gamma,$$

and $F(\cdot + \sqrt{-1}0) - u|_{\Omega_1} \in \mathcal{E}^*(\Omega_1)$.

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