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A THEOREM OF THE ALTERNATIVE IN BANACH LATTICES

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Abstract. We consider a linear sytem in a Banach lattice and provide a simple theorem of the alternative (or Farkas lemma) without the usual closure condition.

1. Introduction

Let X, Z be Banach spaces, with respective topological duals X^*, Z^*, and $A : X^* \to Z^*$ a linear mapping. We consider the existence of solutions to the linear system $\{Ay = b\}$ for some $b \in Z^*$ and when X^* is a Banach lattice for some partial ordering $\leq$.

The usual way to get necessary and sufficient conditions is via a generalized Farkas Lemma (or Theorem of the Alternative) when y is constrained to belong to some convex cone S (the reader is referred to e.g. [3], [5] and the references therein). In [3] a crucial condition is the closure of $A(S)$ in some weak topology, rarely met in practice. In [5] one avoids this closure assumption via an augmented system when the space satisfies some topological assumptions.

We first treat the case where the convex cone S is simply the whole space X^* and therefore the closure condition on $A(X^*)$ is not satisfied. We assume that $I - A$ is not a compact operator, in which case the Fredholm alternative theorem answers the question.

In many applications of interest, X^* has the structure of a Banach lattice for a natural partial ordering (e.g. the L_p spaces for the Volterra and Fredholm type equations or the Poisson equation). We show that this particular structure permits us to derive a simple necessary and sufficient condition for existence of solutions. Incidently, this result also provides a necessary and sufficient condition for existence of a solution *dominated* by some pre-specified element $y_0 \in X^*$. The result simplifies even more if X^* has a *unit* vector I.

Finally, we also consider the existence of *nonnegative* solutions to $\{Ay = b\}$, i.e. when the solution y must be in the (convex) positive cone induced by the partial ordering. Examples of Banach lattices are also provided.

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2. Notation and definitions

Let X be a separable Banach space with a partial ordering $\leq$ such that the positive cone S (i.e. $x \leq y \Leftrightarrow y - x \in S$) is convex and strongly closed. X is also assumed to be a vector lattice for this ordering.

Let X^* be the topological dual of X and $\leq$ the partial ordering on X^* defined as $x \leq y \Leftrightarrow y - x \in S^*$, where $S^* \subset X^*$ is the dual cone of S.

$(X^*, \leq)$ is assumed to be a Banach lattice for this partial ordering, i.e. (i) X^* is a vector lattice ($x \leq y \Rightarrow x + z \leq y + z$), and (ii) if we denote by $|x|$ the absolute value of x ($|x| = (x) \vee (-x),\ x \in X^*$), then

$$|x| \leq |y| \Rightarrow ||x|| \leq ||y||, \qquad x, y \in X^*. \tag{2.1}$$

Note that if x^+ (resp. x^-) is the usual notation for $x \vee 0$ (resp. $x \wedge 0$) then $|x| = x^+ - x^-$.

Convergence in the weak* topology in X^* is denoted by $\stackrel{w*}{\to}$.

Now, let Z^* be the topological dual of a separable Banach space Z (e.g. $Z := X$), and $A : X^* \to Z^*$ a linear mapping. Both (X^*, X) and (Z^*, Z) form a *dual* pair of vector spaces (see e.g. [3]) and the adjoint $A^* : Z \to X$ is given by

$$\langle y, A^* z \rangle = \langle Ay, z \rangle \ \forall y \in X^*,\ z \in Z$$

where $\langle ., . \rangle$ is the duality bracket for a dual pair. Convergence in Z^* for the weak* topology is also denoted by $t*$.

Consider the linear system:

$$Ay = b,\ y \in X^*, \tag{2.2}$$

for some $b \in Z^*$.

If X^* is finite dimensional, then (2.2) has a solution if and only if

$$\forall x \in Z,\ A^* x = 0 \Rightarrow \langle b, x \rangle = 0.$$

The same result (the Fredholm alternative) is true if $I - A$ is a *compact* operator $X^* \to X^*$.

If the solution to (2.2) is constrained to be in a convex cone S^* (the dual cone of a strongly closed convex cone $S \subset X$), then, if A is continuous and $A(S^*)$ is closed (in the weak topologies induced by the dual pairs (X^*, X) and (Z^*, Z)), (2.2) has a solution if and ony if

$$\forall z \in Z,\ A^* z \in S \Rightarrow \langle b, z \rangle \geq 0.$$

This is the Generalized Farkas Theorem of Craven and Koliha, and the reader is referred to [3] for various results along these lines.

However, the condition $A(S^*)$ closed is crucial and rarely met in practice (even in finite dimensional spaces when the cone is not polyhedral). In [5], using an augmented system, we derived a Farkas lemma without this closure assumption.

The idea here is to also consider an augmented system, but we now use the Banach lattice property. That yields a particularly simple Farkas lemma.

3. Main results

The system (2.2) has a solution $y \in X^*$ if and only if the system

$$\left\{ \begin{array}{rcl} A(y_1 - y_2) & = & b, \\ y_1 + y_2 & \leq & y_0, \\ y_1, y_2 & \in & S^*, \end{array} \right. \tag{3.1}$$

has a solution for some fixed $y_0 \in X^*$ (note that, necessarily, $y_0 \in S^*$).

Indeed, the *if* part is trivial, and if y is a solution, then (y_1, y_2) with $y_1 := y^+$ and $y_2 := -y^-$ is a solution to the above system for any $y_0 \geq |y|$.

As we show below, the above linear system (3.1) ensures the conditions that permit us to apply the Generalized Farkas Theorem of Craven and Koliha [3]. We begin with

Theorem 3.1. *Let $A : X^* \to Z^*$ be such that $A^*(Z) \subseteq X$. Then the following two propositons are equivalent:*

(a) $Ay = b$ has a solution $y \in X^*$.

(b) For some $y_0 \in S^*$, $|\langle b, z\rangle| \leq \langle y_0, |A^*z|\rangle$ $\forall z \in Z$.

Proof. It suffices to consider (3.1). Consider the linear mapping $T : (X^*)^3 \to Z^* \times X^*$ with

$$T(y_1, y_2, y_3) = \begin{bmatrix} Ay_1 - Ay_2 \\ y_1 + y_2 + y_3 \end{bmatrix},$$

with adjoint $T^* : Z \times X \to X^3$ given by

$$T^*(z, x) = \begin{bmatrix} A^*z + x \\ -A^*z + x \\ x \end{bmatrix}, \; z \in Z, \; x \in X. \tag{3.2}$$

(3.1) has a solution if and only if

$$T(y_1, y_2, y_3) = \begin{bmatrix} b \\ y_0 \end{bmatrix} \tag{3.3}$$

has a solution $(y_1, y_2, y_3) \in (S^*)^3$.

We first prove that $T(S^3)$ is closed in the space $Z^* \times X^*$ for the weak topology $\sigma(Z^* \times X^*, Z \times X)$ induced by the dual pair $(Z^* \times X^*, Z \times X)$, i.e. for the weak* topology.

The weak* closure of a convex set in the dual of a separable Banach space can be characterized via converging sequences (see e.g. [4]).

Therefore, as $T((S^*)^3) \subset Z^* \times X^*$ is convex, let $\{(y_1^n, y_2^n, y_3^n)\}$ be a sequence in $(S^*)^3$ such that $T(y_1^n, y_2^n, y_3^n)$ converges (weakly *) to some $(a_1, a_2) \in Z^* \times X^*$. We prove that $T(y_1, y_2, y_3) = (a_1, a_2)$ for some $(y_1, y_2, y_3) \in (S^*)^3$.

The weak convergence of $T(y_1^n, y_2^n, y_3^n)$ yields, in particular,

$$y_1^n + y_2^n + y_3^n \overset{w*}{\to} a_2 \tag{3.4}$$

and

$$A(y_1^n - y_2^n) \overset{w*}{\to} a_1. \tag{3.5}$$

Therefore (see e.g. [2]),

$$\sup_n ||y_1^n + y_2^n + y_3^n|| < M \tag{3.6}$$

for some $M > 0$. Moreover, as $y_i^n \in S^*$, $|y_i^n| = y_i^n$. In addition,

$$|y_i^n| \leq |y_1^n + y_2^n + y_3^n| \; \forall i,$$

so that from (2.1) and (3.6) we have

$$\sup_n ||y_i^n|| \leq \sup_n ||y_1^n + y_2^n + y_3^n|| \leq M, \; i = 1, 2, 3. \tag{3.7}$$

As the unit ball in X^* is weak* sequentially compact (see e.g. [2]), there is a subsequence $\{(y_1^{n_k}, y_2^{n_k}, y_3^{n_k})\}$ such that

$$y_i^{n_k} \stackrel{w*}{\to} y_i, \qquad i = 1, 2, 3 \text{ as } k \to \infty. \tag{3.8}$$

As S^* is closed in the weak* topology, $y_i \in S^*\ \forall i$. In addition, as $A^*(Z) \subseteq X$, the mapping A is weakly continuous for the $[\sigma((X^*)^3, X^3), \sigma(Z^* \times X^*, Z \times X)]$ topologies, and thus $A(y_1^{n_k} - y_2^{n_k}) \stackrel{w*}{\to} A(y_1 - y_2)$. Hence, from (3.5), $A(y_1 - y_2) = a_1$.

Having proved that $T((S^*)^3)$ is (weak*) closed, we can apply the Generalized Farkas Theorem of Craven and Koliha ([3], Theorem 2, p. 987, and Theorem 6, p. 989), according to which $T(y_1, y_2, y_3) = (b, y_0)$ has a solution in $(S^*)^3$ if and only if for every $(z, x) \in Z \times X$ that satisfy

$$T^*(z, x) \in S^3$$

then $\langle b, z\rangle + \langle y_0, x\rangle \geq 0$. Indeed, the *dual* cone in X of S^* is $(S^*)^+ := \{x \in X | \langle y, x\rangle \geq 0\ \forall y \in S^*\} = S$, since S is strongly closed (see e.g. [3]).

In view of (3.2), this yields: for every $(z, x) \in Z \times X$ with

$$A^*z + x \in S,\ -A^*z + x \in S,\ x \in S,$$

we have $\langle b, z\rangle + \langle y_0, x\rangle \geq 0$. As $y_0 \in S^*$ and since we must have $x \geq A^*z$ and $x \geq -A^*z$, it suffices to check the above condition for $x := A^*z \vee (-A^*z)$, i.e. $x := |A^*z| \in S$, which yields $\langle b, z\rangle + \langle y_0, |A^*z|\rangle \geq 0$. Now, the same condition for $-z$ yields $\langle b, -z\rangle + \langle y_0, |A^*z|\rangle \geq 0$. Hence, we must have

$$|\langle b, z\rangle| \leq \langle y_0, |A^*z|\rangle\ \forall z \in Z,$$

the desired result. □

Therefore, checking whether (2.2) has a solution reduces to checking (b) for some y_0 that has to be guessed. Equivalently, checking whether (2.2) has no solution reduces to checking whether, for every $y_0 \in S^*$, $|\langle b, z\rangle| > \langle y_0, |A^*z|\rangle$ for some $z \in Z$.

Incidently, for a given $y_0 \in S^*$, (b) also gives a necesary and sufficient condition for the existence of a solution y such that $|y|$ is *dominated* by y_0, which is of interest in some applications.

An important particular case is when the Banach lattice X^* has a *unit* vector I with the property (see e.g. [6])

$$I > 0 \ \text{ and } \forall y \in X^*,\ \exists \alpha > 0 \ \text{ such that } \ -\alpha I \leq y \leq \alpha I. \tag{3.9}$$

In this case we can use $y_0 := \gamma I$ in (3.1) to obtain, finally,

Corollary 3.2. *Let $A : X^* \to Z^*$ be such that $A^*(Z) \subseteq X$. Then the following two propositons are equivalent:*
(a) $Ay = b$ has a solution $y \in X^*$.
(b) For some $\gamma > 0$, $|\langle b, z\rangle| \leq \gamma\langle I, |A^*z|\rangle\ \forall z \in Z$.

4. Cone constrained linear system

We now consider the same linear system $\{Ay = b\}$, but we now require the solution y to be in S^*, i.e. one requires a *nonnegative* solution. In this case we get:

Theorem 4.1. *Let $A : X^* \to Z^*$ be such that $A^*(Z) \subseteq X$. Then the following two propositons are equivalent:*

(a) $Ay = b$ has a solution $y \in S^*$.

(b) For some $y_0 \in S^*$, $\langle y_0, (A^*z)^- \rangle \leq \langle b, z \rangle \leq \langle y_0, (A^*z)^+ \rangle$ $\forall z \in Z$.

In addition, if X^* has a unit vector I, then (b) becomes

(b′) For some $\gamma > 0$, $\gamma\langle I, (A^*z)^- \rangle \leq \langle b, z \rangle \leq \gamma\langle I, (A^*z)^+ \rangle$ $\forall z \in Z$.

Proof. The proof is along the same lines as for Theorem 3.1, except now we consider the linear mapping $T : (X^*)^2 \to Z^* \times X^*$ defined as:

$$T(y_1, y_2) = \begin{bmatrix} Ay_1 \\ y_1 + y_2 \end{bmatrix}.$$

Indeed, the system $\{Ay = b\}$ has a solution $y \in S^*$ if and only if $\{Ay = b,\ 0 \leq y \leq y_0\}$ has a solution for some $y_0 \in S^*$. As in the proof of Theorem 3.1 we can prove that $T((S^*)^2)$ is (weak*) closed, so that again we can apply the Generalized Farkas Theorem of Craven and Koliha [3], which yields

$$z \in Z,\ x \in S,\ A^*z + x \in S \Rightarrow \langle b, z \rangle + \langle y_0, x \rangle \geq 0. \tag{4.1}$$

In (4.1), since $x \geq -A^*z$ and $x \in S$, it suffices to consider $x := (-A^*z) \vee 0 = -(A^*z)^-$, which yields $\langle b, z \rangle \geq \langle y_0, (A^*z)^- \rangle$. A similar argument with $-z$ yields $\langle b, z \rangle \leq \langle y_0, (A^*z)^+ \rangle$, i.e. (b).

If X^* has a unit vector I, then just replace y_0 by γI for some scalar $\gamma > 0$. □

Examples. Below is a list of some Banach lattices.

- $X := R^{n(n+1)/2} =: X^*$, identified with the space of real-symmetric (n, n) matrices with the partial ordering $x \leq y \Leftrightarrow y - x$ positive semi-definite. In this case, X^* has finite dimension but the cone S^* is <u>not</u> polyhedral, so that the standard Farkas lemma does not apply. The identity matrix I is a unit vector.
- $X^* := L^p(\Omega, \mathcal{B}, \mu)$ (with $1 < p \leq \infty$) is a Banach lattice for the natural partial ordering $x \leq y \Leftrightarrow x(\omega) \leq y(\omega),\ \omega \in \Omega$.
- $X^* := l_p$ (with $1 \leq p \leq \infty$). When $X^* := l_1$ then $X := c_0$, the space of sequences that vanish at infinity.
- $X := C_0(\Omega)$, the space of continuous functions on Ω that vanish at infinity (with Ω a separable locally compact Hausdorff space). Then $X^* := M(\Omega)$, the space of finite Borel signed measures on $(\Omega, \mathcal{B})$, and the partial ordering $\mu \leq \nu$ is also the natural ordering $\mu(B) \leq \nu(B),\ B \in \mathcal{B}$.

For examples of particular applications of interest, let us just mention Lyapunov and Riccati equations (or inequalities) for positive semi-definite matrices, Volterra and Fredholm type equations in L_p spaces, as well as the Poisson equation.

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