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ON VAN MILL'S EXAMPLE OF A NORMED X WITH $X \not\approx X \times \mathbb{R}$

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Abstract. In 1987 van Mill constructed an infinite-dimensional normed space X which is not homeomorphic with the product $X \times \mathbb{R}$. We give a short proof of this property of van Mill's example.

In [vM] van Mill has constructed, for every infinite-dimensional separable Banach space E, a dense linear subspace $X \subseteq E$ such that X is not homeomorphic to $X \times \mathbb{R}$. In fact, it is known that the argument from [vM] works for every infinite-dimensional separable F-space E (i.e. a complete metric linear space). The aim of this note is to give a short proof of this property of van Mill's example X. However, our argument does not give the stronger property of the space X — the restricted domain invariance (see [vM, Sec. 4]). The proof uses Mycielski's theorem on independent Cantor sets (see [My, Thm. 1], [Ku, Cor. 3], or [Ke, p. 129] for a short proof of this theorem). Our reasoning is based on the following lemma:

Lemma. *Let E be an infinite-dimensional F-space and let $h = (h_1, h_2) : A \to B$ be a homeomorphism between G_δ-subsets A and B of E and $E \times \mathbb{R}$, respectively. If A contains a dense linear subspace X of E such that $h(X) = X \times \mathbb{R}$, then A contains a topological copy C of the Cantor set with the following properties:*

- (a) *h_1 is injective on C and $C \cap h_1(C) = \emptyset$,*
- (b) *$C \cup h_1(C)$ is linearly independent.*

Proof. It is clear that A is a dense-in-itself completely metrizable space. First, we will define a sequence R_n of closed, nowhere dense relations on A, for $n \geq 1$. Let

$$R_n = \{(x_1, \dots, x_n) \in A^n : (x_1, \dots, x_n, h_1(x_1), \dots, h_1(x_n)) \text{ is linearly dependent}\}$$

for $n \geq 1$. The relation R_n is closed in A^n. We will prove that R_n is nowhere dense in A^n by showing that $X^n \setminus R_n$ is dense in A^n for every $n \geq 1$. We will verify this by induction on n.

Suppose that either $X^{n-1} \setminus R_{n-1}$ is dense in A^{n-1}, or $n = 1$. Fix a sequence $U_1, \dots, U_n$ of nonempty open subsets of E. We shall find $(x_1, \dots, x_n) \in X^n \setminus R_n$ such that $x_i \in U_i$ for $i = 1, \dots, n$.

First, we take $(x_1, \dots, x_{n-1}) \in X^{n-1} \setminus R_{n-1}$ with $x_i \in U_i$ for $i = 1, \dots, n-1$.

Let $F = \operatorname{span}\{x_1, \dots, x_{n-1}, h_1(x_1), \dots, h_1(x_{n-1})\} \subseteq X$ ($F = \{0\}$ if $n = 1$). We have $\dim F = 2n - 2$. Shrinking U_n, if necessary, we may assume that $U_n \cap F = \emptyset$.

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We take any $x \in U_n \cap X$ and we put $G = \operatorname{span}(\{x\} \cup F) \subseteq X$. We have two possibilities:

Case 1. $h(U_n \cap X) \cap (G \times \mathbb{R}) = \emptyset$.

Then we define $x_n = x$.

Case 2. $h(U_n \cap X) \cap (G \times \mathbb{R}) \neq \emptyset$.

In this case $h(U_n \cap X) \cap (G \times \mathbb{R})$ is an open subset of the $2n$-dimensional space $G \times \mathbb{R}$. Hence $[h(U_n \cap X) \cap (G \times \mathbb{R})] \setminus (F \times \mathbb{R})$ is also $2n$-dimensional. Since G is $(2n-1)$-dimensional, we can find $x_n \in (U_n \cap X) \setminus G$ such that $h(x_n) \in (G \times \mathbb{R}) \setminus (F \times \mathbb{R})$, i.e. $h_1(x_n) \in G \setminus F$.

One can easily verify that in both cases the vectors $x_1, \ldots, x_n$ have the required property, which ends the inductive proof of the property of R_n.

Now, we can apply Mycielski's theorem to find a subset C of A homeomorphic to the Cantor set and independent with respect to all relations R_n, $n \geq 1$. This means that for every sequence $x_1, \ldots, x_n$ of distinct elements of C we have $(x_1, \ldots, x_n) \notin R_n$, which gives the required properties (a) and (b) of C. □

The construction of van Mill's example X (see [vM, Thm. 2.1])**.** Let E be an infinite-dimensional separable F-space and let $\mathcal{K}(E)$ be the family of all homeomorphisms $h: C \to D$ between disjoint Cantor sets in E such that the set $C \cup D$ is linearly independent. The family $\mathcal{K}(E)$ has the cardinality of the continuum; therefore we may enumerate it as $\{h_\alpha : C_\alpha \to D_\alpha : \alpha < 2^\omega\}$. Let A be a countable dense subset of E.

By transfinite induction we will choose vectors $x_\alpha, y_\alpha \in E$, for $\alpha < 2^\omega$, such that:

(i) $y_\beta \notin \operatorname{span}(\{x_\gamma : \gamma \leq \alpha\} \cup A)$ for $\beta \leq \alpha$,
(ii) $x_\alpha \in C_\alpha$ and $h_\alpha(x_\alpha) = y_\alpha$.

Suppose that we have chosen x_β and y_β for $\beta < \alpha < 2^\omega$. Let $Y_\alpha = \operatorname{span}(\{x_\beta, y_\beta : \beta < \alpha\} \cup A)$. The linear dimension of the space Y_α is less than 2^ω; hence there exists $c \in C_\alpha$ such that $\operatorname{span}\{c, h_\alpha(c)\} \cap Y_\alpha = \{0\}$. Otherwise, for every $x \in C_\alpha$, we would have that some nonzero $z_x \in Y_\alpha$ is a linear combination of vectors x and $h_\alpha(x)$. Since $C_\alpha \in \mathcal{K}(E)$ the set $\{z_x : x \in C_\alpha\} \subseteq Y_\alpha$ would be linearly independent and of the cardinality of the continuum — a contradiction. We take $x_\alpha = c$ and $y_\alpha = h_\alpha(c)$.

Finally, we define $X = \operatorname{span}(\{x_\alpha : \alpha < 2^\omega\} \cup A)$. The construction of X guarantees that X is dense in E and

$$(\forall h : C \to D \in \mathcal{K}(E))(\exists x \in C \cap X)\ [h(x) \notin X]\,. \tag{1}$$

This implies the following:

The space X **is not homeomorphic with** $X \times \mathbb{R}$**.** Suppose that there exists a homeomorphism g of X onto $X \times \mathbb{R}$. By the Lavrentiev theorem, we can extend g to a homeomorphism $g' = (g'_1, g'_2)$ between G_δ-subsets A and B of E and $E \times \mathbb{R}$ containing X and $X \times \mathbb{R}$, respectively. From our Lemma it follows that there is an $h \in \mathcal{K}(E)$ such that g'_1 is an extension of h. This contradicts the property (1) of X.

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