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COUNTABLE LINEAR TRANSFORMATIONS ARE CLEAN

W. K. NICHOLSON AND K. VARADARAJAN

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ABSTRACT. It is shown that every linear transformation on a vector space of countable dimension is the sum of a unit and an idempotent.

An element in a ring R is called *clean* in R if it is the sum of a unit and an idempotent, and the ring itself is called *clean* if every element is clean. Every clean ring is an exchange ring and, if R has central idempotents, R is an exchange ring if and only if it is clean [2, Proposition 1.8]. Camillo and Yu [1, Theorem 9] have shown that a ring is semiperfect if and only if it is clean and has no infinite orthogonal family of idempotents.

Our main result is the following theorem which answers a question of P. Ara.

Theorem. *If V_D is a vector space of countably infinite dimension over a division ring D, then* $\operatorname{end}(V_D)$ *is clean.*

A ring R is called *unit regular* if, for each $a \in R$, there exists a unit $u \in R$ such that $aua = a$. Camillo and Yu [1, Theorem 5] show that every unit regular ring is clean. The Theorem shows that the converse is not true.

Corollary. *There exists a* (*von Neumann*) *regular, right self-injective, clean ring which is not unit regular.*

Proof. The ring $\operatorname{end}(V_D)$ in the Theorem suffices because it is not unit regular. In fact, it is not even Dedekind finite ($ab = 1$ implies $ba = 1$). □

The proof of the Theorem employs several preliminary lemmas. Throughout this paper D always denotes a division ring and V_D is always a vector space of countably infinite dimension over D. If $\{x_1, x_2, \dots\}$ is a basis of V_D, the linear transformation $\sigma\colon V \to V$ given by $\sigma(x_i) = x_{i+1}$ for each i is called a *shift operator* on V.

Lemma 1. *Every shift operator on V_D is clean in* $\operatorname{end}(V_D)$.

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Proof. Let $\sigma\colon V \to V$ be the shift operator relative to the basis $\{x_1, x_2, \dots\}$ of V_D. Define $\pi\colon V \to V$ as follows:

$$\begin{aligned}
\pi(x_1) &= x_1 + x_2, \\
\pi(x_2) &= 0, \\
\pi(x_{2k+1}) &= x_{2k} + x_{2k+2} && \text{for } k \geq 1, \\
\pi(x_{4k}) &= x_{4k} + x_2 && \text{for } k \geq 1, \\
\pi(x_{4k+2}) &= x_{4k+2} - x_2 && \text{for } k \geq 1.
\end{aligned}$$

Then it is a routine matter to check that $\pi^2(x_i) = \pi(x_i)$ for each i, so that π is an idempotent in $\operatorname{end}(V_D)$. The action of $\sigma - \pi$ on the basis $\{x_1, x_2, \dots\}$ is:

$$\begin{aligned}
(\sigma - \pi)(x_1) &= -\, x_1, \\
(\sigma - \pi)(x_2) &= x_3, \\
(\sigma - \pi)(x_{2k+1}) &= -\, x_{2k} && \text{for } k \geq 1, \\
(\sigma - \pi)(x_{4k}) &= x_{4k+1} - x_{4k} - x_2 && \text{for } k \geq 1, \\
(\sigma - \pi)(x_{4k+2}) &= x_{4k+3} - x_{4k+2} + x_2 && \text{for } k \geq 1.
\end{aligned}$$

It follows that x_{2k} is in $\operatorname{im}(\sigma - \pi)$ for each $k \geq 1$, and hence that x_{4k+1} and x_{4k+3} are in $\operatorname{im}(\sigma - \pi)$. Hence $\sigma - \pi$ is onto. Finally, if $(\sigma - \pi)(x_1 a_1 + x_2 a_2 + \cdots) = 0$, a_i in D, we find

$$\begin{aligned}
&\text{coefficient of } x_1\colon && -\, a_1, \\
&\text{coefficient of } x_2\colon && -\, a_3 - a_4 + a_6 - a_8 + a_{10} - \cdots, \\
&\text{coefficient of } x_{2k+1}\colon && a_{2k} && \text{for } k \geq 1, \\
&\text{coefficient of } x_{2k+2}\colon && -\, a_{2k+2} - a_{2k+3} && \text{for } k \geq 1.
\end{aligned}$$

Since all these coefficients vanish, we have $a_k = 0$ for each k. Thus $\sigma - \pi$ is one-to-one. □

We denote the ring of $n \times n$ matrices over a ring R by $M_n(R)$.

Lemma 2. *If R is a clean ring, then any $n \times n$ companion matrix over R is clean in $M_n(R)$.*

Proof. Such a matrix has the form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_1 \\ 1 & 0 & 0 & \dots & 0 & a_2 \\ 0 & 1 & 0 & \dots & 0 & a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & e \\ 0 & 0 & 0 & \dots & 0 & e \\ 0 & 0 & 0 & \dots & 0 & e \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & e \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & u \\ 1 & 0 & 0 & \dots & 0 & b_2 \\ 0 & 1 & 0 & \dots & 0 & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_n \end{bmatrix}$$

where $a_1 = e + u$ with $e^2 = e$ and u is invertible in R. The first of these matrices is an idempotent in the matrix ring $M_n(R)$, and the second matrix is invertible. □

Lemma 3. *If $\alpha \in \operatorname{end}(V_D)$ is such that V is spanned by $\{y, \alpha(y), \alpha^2(y), \dots\}$ for some $y \in V$, then α is clean in* $\operatorname{end}(V_D)$.

Proof. We may assume that $V \neq 0$. If $\alpha^n(y) \notin yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$ for all $n \geq 1$, then $\{y, \alpha(y), \alpha^2(y), \dots\}$ is a basis of V. Since α is the shift operator with respect to this basis, it is clean by Lemma 1.

So assume that $\alpha^n(y) \in yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$ for some $n \geq 1$. If n is minimal with this property, then $\{y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y)\}$ is a basis of V_D. Thus α is clean by Lemma 2 because its matrix with respect to this basis is a companion matrix. □

Lemma 4. *Let $\alpha \in \operatorname{end}(V)$ and let U be an α-invariant subspace of V. Assume that a vector $y \in V - U$ exists such that $V = U + K$ where $K = yD + \alpha(y)D + \cdots$. If the restriction $\alpha_{|U}$ is clean in $\operatorname{end}(U)$, then α is clean in $\operatorname{end}(V)$. More precisely, if $\alpha_{|U} = \pi + \sigma$ in $\operatorname{end} U$, $\pi^2 = \pi$, σ invertible, then $\alpha = \varphi + \tau$ in $\operatorname{end} V$, $\varphi^2 = \varphi$, τ invertible, where $\varphi_{|U} = \pi$ and $\tau_{|U} = \sigma$.*

Proof. Write $V = M \oplus U$ where M is a subspace containing y. For convenience write $\overline{v} = v + U$ for each $v \in V$. Since U is α-invariant, we get $\tilde{\alpha}\colon V/U \to V/U$ defined by $\tilde{\alpha}(\overline{v}) = \overline{\alpha(v)}$. It follows by induction that

$$\overline{\alpha^n(v)} = \tilde{\alpha}^n(\overline{v}) \quad \text{for all } v \in V \text{ and all } n \geq 1. \tag{0.1}$$

Now let $\theta^2 = \theta \in \operatorname{end}(V)$ satisfy $\theta(V) = M$ and $\ker(\theta) = U$. Then θ induces a D-isomorphism $\theta_0\colon V/U \to M$ given by $\theta_0(\overline{v}) = \theta(v)$ for all $v \in V$. Hence we have

$$M \xrightarrow{\theta_0^{-1}} V/U \xrightarrow{\tilde{\alpha}} V/U \xrightarrow{\theta_0} M$$

and we write $\beta = \theta_0\tilde{\alpha}\theta_0^{-1} \in \operatorname{end}(M)$. Thus $\beta\theta_0 = \theta_0\tilde{\alpha}$ and one verifies that

$$\beta[\theta(v)] = \theta[\alpha(v)] \quad \text{for all } v \in V. \tag{0.2}$$

If $m \in M$, this gives $\theta[\alpha(m)] = \beta[\theta(m)] = \beta(m)$ because $m \in M = \operatorname{im}(\theta)$. Thus $\theta[\beta(m)] = \theta^2[\alpha(m)] = \theta[\alpha(m)]$. Since $\ker(\theta) = U$, this proves

$$\alpha(m) - \beta(m) \in U \quad \text{for all } m \in M. \tag{0.3}$$

Our hypothesis guarantees that $\{\overline{y}, \overline{\alpha(y)}, \overline{\alpha^2(y)}, \ldots\}$ spans V/U, so (0.1) shows that $\{\overline{y}, \tilde{\alpha}(\overline{y}), \tilde{\alpha}^2(\overline{y}), \ldots\}$ spans V/U. If we apply θ_0 we find that $\{\theta_0[\overline{y}], \theta_0[\tilde{\alpha}(\overline{y})], \theta_0[\tilde{\alpha}^2(\overline{y})], \ldots\}$ spans M so, by Lemma 3, we have

$$\beta = \sigma_0 + \pi_0 \quad \text{where } \pi_0^2 = \pi_0 \in \operatorname{end}(M) \text{ and } \sigma_0 \in \operatorname{end}(M) \text{ is a unit.}$$

Since $\alpha_{|U}$ is clean in $\operatorname{end}(U)$ by hypothesis, write

$$\alpha_{|U} = \sigma + \pi \quad \text{where } \pi^2 = \pi \in \operatorname{end}(U) \text{ and } \sigma \in \operatorname{end}(U) \text{ is a unit.}$$

Finally, since $V = M \oplus U$, define φ and τ in $\operatorname{end}(V)$ by

$$\begin{aligned} \varphi(m+u) &= \pi_0(m) + \pi(u), \\ \tau(m+u) &= \sigma_0(m) + [\alpha(m) - \beta(m) + \sigma(u)] \end{aligned}$$

where we note (0.3) in the definition of τ. Clearly $\varphi_{|U} = \pi$ and $\tau_{|U} = \sigma$. Since $\sigma_0 + \pi_0 = \beta$ and $\sigma + \pi = \alpha_{|U}$, we have $\alpha = \varphi + \tau$ because

$$\begin{aligned} (\varphi + \tau)(m+u) &= [\pi_0(m) + \sigma_0(m)] + [\pi(u) + \sigma(u)] + [\alpha(m) - \beta(m)] \\ &= \beta(m) + \alpha(u) + [\alpha(m) - \beta(m)] \\ &= \alpha(u+m). \end{aligned}$$

We have $\varphi^2 = \varphi$ because $\pi_0^2 = \pi_0$ and $\pi^2 = \pi$, so it remains to show that τ is an automorphism of V. It is monic because $\tau(m+u) = 0$ implies $\sigma_0(m) = 0$ and $\alpha(m) - \beta(m) + \sigma(u) = 0$, whence $m = 0 = u$. To show that τ is epic, observe first that $U \subseteq \operatorname{im}(\tau)$ because, if $u \in U$, $u = \sigma(u_0) = \tau(0 + u_0)$ for some $u_0 \in U$. So it remains to show that $M \subseteq \operatorname{im}(\tau)$. If $m \in M$ write $m = \sigma_0(m_1)$, $m_1 \in M$, and then write $\alpha(m_1) - \beta(m_1) = -\sigma(u_1)$, $u_1 \in U$. Then

$$\tau(m_1 + u_1) = \sigma_0(m_1) + [\alpha(m_1) - \beta(m_1) + \sigma(u_1)] = m.$$

Thus $M \subseteq \operatorname{im}(\tau)$ and the proof is complete. □

Proof of the Theorem. Fix α in $\operatorname{end}(V)$ and define

$$\mathcal{S} = \{(U, \sigma, \pi) | U_D \subseteq V \text{ is } \alpha\text{-invariant}, \\ \alpha_{|U} = \sigma + \pi, \sigma \in \operatorname{end}(U) \text{ is a unit and } \pi^2 = \pi \in \operatorname{end}(U)\}.$$

Then $(0, 0, 0) \in \mathcal{S}$. Partially order $\mathcal{S}$ by writing $(U, \sigma, \pi) \leq (U', \sigma', \pi')$ if $U \subseteq U'$, $\sigma = \sigma'_{|U}$ and $\tau = \tau'_{|U}$. This is inductive so, by Zorn's lemma, let (U, σ, π) be maximal in $\mathcal{S}$. It suffices to show that $U = V$. If not, choose $y \in V - U$, let $K = yD + \alpha(y)D + \alpha^2(y)D + \cdots$ and write $V_0 = U + K$. Then V_0 and K are α-invariant and, regarding $\alpha \in \operatorname{end}(V_0)$, $\alpha_{|U}$ is clean in $\operatorname{end}(U)$ because $(U, \sigma, \pi) \in \mathcal{S}$. Hence α is clean in $\operatorname{end}(V_0)$ by Lemma 4, contradicting the maximality of $(U, \sigma, \pi) \in \mathcal{S}$. This proves the Theorem. □

The ring $\operatorname{end}(V_D)$ in the Theorem can be regarded as the ring of column finite, countably infinite square matrices over the division ring D. An interesting related question (also due to P. Ara) is the following:

Question 1. Is the ring of countably infinite, row and column finite matrices over a division ring clean?

This would be true, for example, if the idempotents constructed in the proof of the theorem were row and column finite. (This would show that any subring of $\operatorname{end}(V)$ containing the row and column finite matrices would be clean.) However, the idempotent π constructed in Lemma 1 for the shift operator is not row finite.

Question 2. Does the Theorem remain true for vector spaces of arbitrary infinite dimension over a division ring?

References

[1] V. P. Camillo and H.-P. Yu, *Exchange rings, units and idempotents*, Comm. Alg. **22** (1994), 4737–4749. MR **95d:**16013

[2] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. A.M.S. **229** (1977), 269–278. MR **55:**12757

Department of Mathematics, University of Calgary, Calgary, Canada T2N 1N4
E-mail address: wknichol@acs.ucalgary.ca

Department of Mathematics, University of Calgary, Calgary, Canada T2N 1N4
E-mail address: varadara@math.ucalgary.ca