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VAN DOUWEN'S PROBLEM ON 0-DIMENSIONAL IMAGES OF ORDERED COMPACTA

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ABSTRACT. A compact 0-dimensional space has a T_0-separating rank 1 family of clopen sets iff it is the continuous image of a compact 0-dimensional linearly ordered space.

The claim and conjecture below were made by E. K. van Douwen. It will be shown that the claim is true and the conjecture is false. (Also see [P].) Translated into the language of Boolean algebras, this solves Problem 1 in [KM]: any subalgebra of an interval algebra is isomorphic to a pseudo-tree algebra.

A family of sets is *rank* 1 if any two of its members are comparable or disjoint. A family of subsets of a set X is T_0-*separating* if for any two distinct points of X, there is a member of the family containing one of these points but not the other point.

Claim ([D]). If a compact space has a T_0-separating rank 1 family of clopen sets, then it is the continuous image of a compact 0-dimensional linearly ordered space.

It is unknown whether van Douwen ever wrote down a proof of his claim.

Conjecture ([D]). For 0-dimensional spaces the converse of the above claim is false.

A *dendron* D is a continuum such that for any distinct x and y in D there exists $d \in D$ such that x and y are in different components of $D - \{d\}$. A family of subsets of a set is *cross-free* if any two of its members are comparable or disjoint or their union is X.

Lemma 1 ([MW]). *A Hausdorff space X can be embedded into a dendron iff X possesses a cross-free closed subbase.*

Lemma 2 ([B] and [C]). *Every dendron is the continuous image of a linearly ordered continuum.*

Theorem 1. *Van Douwen's claim is true.*

Proof. Let $\mathcal{T}$ be a T_0-separating rank 1 family of clopen sets in the space X and $\mathcal{T}'$ be the family of all complements of members of $\mathcal{T}$. Since $\mathcal{T}$ is rank 1, it is easy to see that $\mathcal{S} = \mathcal{T} \cup \mathcal{T}'$ is cross free. In addition $\mathcal{S}$ is a family of clopen sets that separates

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points (i.e., is Hausdorff separating) because $\mathcal{T}$ is a T_0-separating family of clopen sets. Therefore, $\mathcal{S}$ is a subbase for X as a Hausdorff separating family of open sets in a compact space. Hence by Lemma 1, X can be embedded into a dendron. So by Lemma 2, X is the continuous image of an ordered continuum C. But C is the continuous image (under the canonical quotient map) of the compact 0-dimensional linearly ordered space $C \times \{0, 1\}$ with the lexicographic order topology. □

The following result was obtained in collaboration with P. J. Nyikos ([Ny]).

Lemma 3. *If a T_0 space has a cross-free clopen subbase, then it has a rank 1 T_0-separating clopen cover.*

Proof. Let $\mathcal{C}$ be a cross-free clopen subbase of a space X. Consider the cross-free clopen subbase $\mathcal{C}' = \mathcal{C} \cup \{A\colon\ X - A \in \mathcal{C}\}$. Let $\mathcal{M}$ be a maximal rank 1 subcollection of $\mathcal{C}'$.

Suppose $\mathcal{M}$ is not T_0-separating, say $\{p, q\}$ is not separated. Since $\mathcal{C}'$ is a subbase, there is a member S which (without loss of generality) contains p but not q. By hypothesis, there exists $M \in \mathcal{M}$ such that $M \cup S = X$, but yet $M \neq X$ (otherwise we could just add S to $\mathcal{M}$, contradicting maximality).

Now $X - S$ separates p from q, and so $X - S$ can't be in $\mathcal{M}$ either. Moreover, there must be $M' \in \mathcal{M}$ such that $(X - S) \cup M' = X$, and yet again M' is not all of X.

But now, both M and M' meet $\{p, q\}$ and so each contains it, so one must contain the other, say $M' \subseteq M$. But now the claim that $(X - S) \cup M' = X$ puts S inside M, contradicting the earlier conclusion that $M \cup S = X$.

The next result appears in a paper containing a theorem for which the author of the present paper presented a counterexample. But a careful check of the proof of the next result has shown it to be correct. □

Lemma 4 ([Ni])**.** *Let X be a Hausdorff 0-dimensional space which is the continuous image of a compact linearly ordered space. Then X can be embedded into a dendron.*

Lemma 5 ([MW])**.** *Let D be a dendron and $\mathcal{U}$ be the collection of all components of $D - \{p\}$ for each $p \in D$. Then $\mathcal{U}$ is a cross-free open subbase for D.*

Theorem 2. *The conjecture is false. Hence a compact 0-dimensional space has a T_0-separating rank 1 family of clopen sets iff it is the continuous image of a compact 0-dimensional linearly ordered space.*

Proof. Nikiel's proof of Lemma 4 indicates that for the constructed dendron D containing X we have: for each pair $\{x, y\}$ of distinct points of X there exists $d \in D - X$ such that x and y are in different components of $D - \{d\}$. So by Lemma 5, $\mathcal{C} = \{C \cap X\colon\ C$ is a component of $D - \{p\},\ p \in D - X\}$ is a cross-free clopen subbase of X since all boundary points of elements of $\mathcal{C}$ are in $D - X$ and X is compact. Hence by Lemma 3, X possesses a rank 1 T_0-separating family of clopen sets. □

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