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PRIMES OF THE FORM $p = 1 + m^2 + n^2$ IN SHORT INTERVALS

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Abstract. In this note, we prove that for every $\theta \geq \frac{115}{121}$ and $x \geq x_0(\theta)$, the short interval $(x, x + x^\theta]$ contains at least one prime number of the form $p = 1 + m^2 + n^2$ with $(m, n) = 1$. This improves a similar result due to Huxley and Iwaniec, which requires $\theta \geq \frac{99}{100}$.

§1. Introduction

The existence of prime numbers in short intervals is an important subject in analytic number theory. Huxley and Iwaniec [3] asked the following question: seek positive numbers θ, as small as possible, such that

(1.1). *the short interval* $(x, x + x^\theta]$ *contains at least one prime number of the form* $p = 1 + m^2 + n^2$ *with* $(m, n) = 1$ *for* $x \geq x_0(\theta)$.

Naturally this problem can be attacked by applying the half dimensional sieve to the sequence

$$\mathcal{A} := \{p - 1 : x < p \leq x + x^\theta,\, p \equiv 3 \,(\mathrm{mod}\, 8)\},$$

where, as in the sequel, the letter p, with or without subscript, denotes a prime number. In order to control error terms in formulas of sieves, a Bombieri–Vinogradov type mean-value theorem for short intervals is needed. Huxley and Iwaniec [3] have obtained a satisfactory generalization of Bombieri–Vinogradov's theorem in the case of short intervals, using a zero-density theorem for Dirichlet L-functions. As an application, they have shown that (1.1) is true with $\theta = \frac{99}{100}$.

The aim of this note is to improve Huxley and Iwaniec's exponent. More precisely we shall prove the following result.

Theorem. *For every* $\theta \geq \frac{115}{121}$ *and* $x \geq x_0(\theta)$, *we have*

$$\sum_{x < p \leq x + x^\theta} b^*(p - 1) \gg x^\theta / (\log x)^{3/2}, \tag{1.2}$$

where

$$b^*(a) := \begin{cases} 1 & \textit{if } a = m^2 + n^2 \textit{ with } (m, n) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

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For comparison, we have $\frac{115}{121} \approx 0.9504$ and $\frac{99}{100} = 0.99$.

We begin in the same way as Huxley and Iwaniec. By (3.2) of [3], we have

$$\sum_{x<p\le x+x^\theta} b^*(p-1) = S(\mathcal{A};\mathcal{P}_3, x+x^\theta), \tag{1.3}$$

where $S(\mathcal{A};\mathcal{P}_3,z) := \big|\big\{a \in \mathcal{A} : (a, \prod_{p<z,\, p\in\mathcal{P}_3} p) = 1\big\}\big|$ and $\mathcal{P}_3 := \{p : p \equiv 3 \pmod 4\}$. Let $\alpha = \alpha(\theta) \in [2,3)$ be a parameter to be chosen later. For $z = x^{1/\alpha}$, we write

$$S(\mathcal{A};\mathcal{P}_3, x+x^\theta) = S(\mathcal{A};\mathcal{P}_3, z) - T, \tag{1.4}$$

where $T := S(\mathcal{A};\mathcal{P}_3,z) - S(\mathcal{A};\mathcal{P}_3,x+x^\theta)$. A lower bound for $S(\mathcal{A};\mathcal{P}_3,z)$ on the right-hand side of (1.4) will be obtained by the half dimensional sieve (as in [3]) in Section 3. We shall give a better upper bound for T than that of Huxley and Iwaniec. This improvement comes from our generalized Bombieri–Vinogradov type mean-value theorem for short intervals that we state in Section 2. One observes that each element $a \in \mathcal{A}$ is divisible by an even number of primes from $\mathcal{P}_3$ and $2 \parallel a$. Hence, for $z = x^{1/\alpha}$ with $2 \le \alpha < 3$, we have obviously

$$T \le \sum_{\substack{x<p\le x+x^\theta \\ p=1+2np_1p_2}} 1, \tag{1.5}$$

where $p_1 \in \mathcal{P}_3$, $p_2 \in \mathcal{P}_3$, $p_1 \ge p_2 \ge x^{1/\alpha}$ and n is an integer divisible only by primes of the form $p \equiv 1 \pmod 4$. We define

$$\begin{aligned}\mathcal{L} := \big\{l = 2np_2 : n \le x^{1-2/\alpha}, p \mid n \Rightarrow p \equiv 1 \pmod 4;& \\ x^{1/\alpha} < p_2 \le (x/n)^{1/2},\ p_2 \in \mathcal{P}_3\big\}&.\end{aligned}$$

For every $l \in \mathcal{L}$, we put

$$\mathcal{M}(l) := \big\{m = lp_1 + 1 : x/l < p_1 \le (x+x^\theta)/l,\ (l/2)p_1 \equiv 1 (\mathrm{mod}\, 4)\big\}.$$

It is clear that the sum on the right-hand side of (1.5) does not exceed the number of primes in the set $\bigcup_{l\in\mathcal{L}} \mathcal{M}(l)$; thus

$$T \le \sum_{l\in\mathcal{L}} \big\{S\big(\mathcal{M}(l);\mathcal{P}(l),(x/l)^{\theta_1}\big) + O\big((x/l)^{\theta_1}\big)\big\}, \tag{1.6}$$

where $\mathcal{P}(l) := \{p : (p,l) = 1\}$ and $\theta_1 := (2\theta-1)/4$. We shall apply an upper bound formula of the linear sieve to treat $S\big(\mathcal{M}(l);\mathcal{P}(l),(x/l)^{\theta_1}\big)$ in Section 4.

It seems interesting to compare the theorem with the following results: for x sufficiently large, $(x, x+x^{0.973}]$ contains at least one prime of the form $p = -2 + P_2$ where P_2 denotes an integer having at most two prime factors [8]; for every $\varepsilon > 0$ and $x \ge x_0(\varepsilon)$, there exists at least one prime of the form $p = -2+a$ in $(x, x+x^{3/4+\varepsilon}]$ where a is a $\mathcal{B}$–free integer [7].

§2. Two preliminary lemmas

We first recall some standard notations (cf. [2]). Let $\mathcal{F}$ be a finite sequence of integers and let $\mathcal{P}$ be a set of prime numbers. For $z \ge 2$, the sifting function is defined as follows:

$$S(\mathcal{F};\mathcal{P},z) := \big|\big\{a \in \mathcal{F} : \big(a, P(z)\big) = 1\big\}\big|,$$

where $P(z) := \prod_{p<z,\, p\in\mathcal{P}} p$. If d is a squarefree integer whose prime factors belong to $\mathcal{P}$, we let $\mathcal{F}_d := \{n \in \mathcal{F} : d|n\}$ and use $|\mathcal{F}_d|$ to denote the number of elements of $\mathcal{F}_d$. We write an approximate formula

$$|\mathcal{F}_d| = \frac{w(d)}{d} X + r(\mathcal{F}, d),$$

where $X > 1$ is independent of d, and $w(d)$ is a multiplicative function. We define

$$V(z) := \prod_{p<z,\, p\in\mathcal{P}} \bigl(1 - w(p)/p\bigr).$$

As usual, $\mu(n)$ is Möbius' function, $\varphi(n)$ Euler's function and $\omega(n)$ the number of distinct prime factors of n. Finally we write ε for an arbitrarily small positive number and γ for Euler's constant.

The first lemma is an upper bound formula of the linear sieve ([2], Theorem 8.3).

Lemma 1. *If there exist positive constants* A_k $(k = 1, 2, 3)$ *such that*

$$0 \le \frac{w(p)}{p} < 1 - \frac{1}{A_1},$$
$$-A_2 \le \sum_{z_1 \le p < z_2} w(p) \frac{\log p}{p} - \log\frac{z_2}{z_1} \le A_3 \qquad (z_2 \ge z_1 \ge 2),$$

we then have, for $2 \le z \le Q$, *that*

$$S(\mathcal{F}; \mathcal{P}, z) \le XV(z)\Bigl\{F\Bigl(\frac{\log Q}{\log z}\Bigr) + O\Bigl(\frac{A_2}{(\log Q)^{1/14}}\Bigr)\Bigr\} + \sum_{d<Q,\, d|P(z)} 3^{\omega(d)} |r(\mathcal{F}, d)|,$$

where $F(t) = 2e^{\gamma}/t$ $(0 < t \le 2)$.

The following lemma is a direct consequence of Theorem 2 of [6].

Lemma 2. *Let* $g(l)$ *be an arithmetic function satisfying* $g(l) \ll 1$ *and let*

$$H(x', h, q, a, l) := \sum_{\substack{x' < lp \le x'+h \\ lp \equiv a \,(\mathrm{mod}\, q)}} - \frac{1}{\varphi(q)} \int_{x'/l}^{(x'+h)/l} \frac{\mathrm{d}t}{\log t}.$$

Then for any $A > 0$, *there exists a positive constant* $B = B(A)$ *such that*

(2.1)
$$\sum_{q \le Q} \max_{(a,q)=1} \max_{h \le x^{\theta}} \max_{x/2 < x' \le x} \Bigl| \sum_{\substack{l \le L \\ (l,q)=1}} g(l) H(x', h, q, a, l) \Bigr| \ll \frac{x^{\theta}}{(\log x)^A},$$

$$\sum_{q \le Q} \mu(q)^2 3^{\omega(q)} \max_{(a,q)=1} \max_{h \le x^{\theta}} \max_{x/2 < x' \le x} \Bigl| \sum_{\substack{l \le L, \\ (l,q)=1}} g(l) H(x', h, q, a, l) \Bigr| \ll \frac{x^{\theta}}{(\log x)^A} \tag{2.2}$$

for $x \ge 10$, $\frac{3}{5} + \varepsilon \le \theta \le 1$, $Q = x^{\theta - 1/2}/(\log x)^B$ *and* $L = x^{(5\theta-3)/2-\varepsilon}$.

§3. Lower bound for $S(\mathcal{A};\mathcal{P}_3,x^{1/\alpha})$

The following proposition offers the required lower bound for $S(\mathcal{A};\mathcal{P}_3,x^{1/\alpha})$.

Proposition 1. *Assume that* $\frac{1}{2}\le\theta\le 1$ *and* $2/(2\theta-1)\le\alpha\le 6/(2\theta-1)$. *We then have*

$$S(\mathcal{A};\mathcal{P}_3,x^{1/\alpha})\ge\{W_1(\theta,\alpha)+o(1)\}\,x^{\theta}/(\log x)^{3/2}, \tag{3.1}$$

where

$$W_1(\theta,\alpha):=\frac{AC_3}{\sqrt{4\theta-2}}\int_1^{\alpha(\theta-1/2)}\frac{\mathrm{d}t}{\sqrt{t(t-1)}}$$

and $A:=\big(1/2\sqrt{2}\big)\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\big(1-p^{-2}\big)^{1/2}$, $C_3:=\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\big(1-(p-1)^{-2}\big)$.

Proof. By (3.3) of [3] or Theorem 1 of [4], we have

$$S(\mathcal{A};\mathcal{P}_3,x^{1/\alpha})\ge\{1+o(1)\}\frac{x^{\theta}V(x^{1/\alpha})}{4\log x}\sqrt{\frac{e^{\gamma}}{\pi\xi}}\int_1^{\xi}\frac{\mathrm{d}t}{\sqrt{t(t-1)}}-E(x,x^{\theta},Q), \tag{3.2}$$

where

$$\xi=\alpha\frac{\log Q}{\log x}\in[1,3],\qquad V(x^{1/\alpha})=\prod_{\substack{p<x^{1/\alpha}\\ p\equiv 3\,(\mathrm{mod}\,4)}}\Big(1-\frac{1}{p-1}\Big),$$

$$E(x,x^{\theta},Q)=\sum_{q\le Q}\max_{(a,q)=1}\max_{h\le x^{\theta}}\max_{x/2<x'\le x}\Bigg|\sum_{\substack{x'<p\le x'+h\\ p\equiv a\,(\mathrm{mod}\,q)}}1-\frac{1}{\varphi(q)}\int_{x'}^{x'+h}\frac{\mathrm{d}t}{\log t}\Bigg|.$$

Let χ be the non-principal character modulo 4 and let $L(\chi,s)$ be the Dirichlet L-function associated with χ. Using the relation $L(\chi,1;y):=\prod_{p<y}(1-\chi(p)/p)^{-1}\to L(\chi,1)=\frac{1}{4}\pi$ $(y\to\infty)$ and Mertens' formula, we deduce that

$$\begin{aligned}V(x^{1/\alpha})&=\sqrt{2L(\chi,1;x^{1/\alpha})}\prod_{\substack{p<x^{1/\alpha}\\ p\equiv 3\,(\mathrm{mod}\,4)}}\big(1-p^{-2}\big)^{1/2}\big(1-(p-1)^{-2}\big)\\&\qquad\times\prod_{p<x^{1/\alpha}}\big(1-p^{-1}\big)^{1/2}\\&=\{1+o(1)\}\,2AC_3(\alpha\pi e^{-\gamma}/\log x)^{1/2}\qquad(x\to\infty).\end{aligned} \tag{3.3}$$

Taking $Q=x^{\theta-1/2}/(\log x)^B$ and $g(1)=1,g(l)=0$ $(l\ge 2)$ in (2.1) of Lemma 2, we obtain

$$E(x,x^{\theta},Q)\ll x^{\theta}/(\log x)^2. \tag{3.4}$$

Now the required inequality (3.1) follows from (3.2)–(3.4). ☐

§4. Upper bound for T

The purpose of this section is to prove Proposition 2 below, which gives a better upper bound for T than that of [3]. To prepare for its proof, we first establish two auxiliary results.

Lemma 3. *Let $u(n)$ be the characteristic function of integers whose prime factors are of the form $4m+1$ and let $f(n) := \prod_{p|n,\, p>2}(p-1)/(p-2)$. We then have*

$$\sum_{n\le x} u(n)f(n) = (A/C_1)x/(\log x)^{1/2} + O\big(x/(\log x)^{3/2}\big), \tag{4.1}$$

where $C_1 := \prod_{p\equiv 1\,(\mathrm{mod}\,4)}\big(1-(p-1)^{-2}\big)$ and A is defined as in Proposition 1.

Proof. Let $\chi, L(s,\chi)$ be defined as above. It is clear that $u(n)f(n)$ is multiplicative and

$$u(p^\nu)f(p^\nu) = \begin{cases} (p-1)/(p-2) & \text{if } p\equiv 1\,(\mathrm{mod}\,4),\\ 0 & \text{otherwise.}\end{cases}$$

A simple calculation shows that for $\Re e\, s > 1$,

$$\sum_{n=1}^{\infty} u(n)f(n)n^{-s} = \prod_{p\equiv 1(\mathrm{mod}\,4)} \big(1-p^{-s}\big)^{-1}\big(1+(p-2)^{-1}p^{-s}\big) = \zeta(s)^{1/2}G(s),$$

where $\zeta(s)$ is Riemann's zeta function and

$$G(s) := \Big(L(s,\chi)(1-2^{-s})\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})\Big)^{1/2}\prod_{p\equiv 1\,(\mathrm{mod}\,4)}\big(1+(p-2)^{-1}p^{-s}\big).$$

Using the well known estimation (see [1], Theorem 8.1)

$$L(s,\chi) \ll \log(3+|\Im m\, s|) \quad \text{for } \Re e\, s \ge 1 - 1/\log(3+|\Im m\, s|),$$

we see that $\sum_{n=1}^{\infty} u(n)f(n)n^{-s}$ is of type $\mathcal{T}(\frac12, \frac12\,; c_0, \delta, M)$, where c_0, δ, M are suitable positive constants (see page 185 of [5] for the definition of $\mathcal{T}(\frac12, \frac12\,; c_0, \delta, M)$). Thus Theorem II.5.3 of [5] is applicable. Now our required result follows immediately from this theorem with $N = 0$. □

Lemma 4. *Let $\mathcal{L}$, $f(n)$, A, C_1 be defined as above. Assume that $2 \le \alpha < 3$. We have*

$$\sum_{l\in\mathcal{L}} \frac{f(l)}{l\log^2(x/l)} = \frac{1+o(1)}{(\log x)^{3/2}}\cdot\frac{A\alpha}{4C_1}\int_2^{\alpha}\frac{t-2+(t-1)\log(t-1)}{t^2(t-1)(1-t/\alpha)^{1/2}}\,\mathrm{d}t. \tag{4.2}$$

Proof. Let Y be the sum on the left-hand side of (4.2) and let $u(n)$ be the function defined as in Lemma 3. We have

$$Y = \frac{1+o(1)}{2}\sum_{n\le x^{1-2/\alpha}}\frac{u(n)f(n)}{n}\sum_{\substack{x^{1/\alpha}<p_2\le(x/n)^{1/2}\\ p_2\equiv 3\,(\mathrm{mod}\,4)}}\frac{1}{p_2\log^2(x/np_2)}.$$

As usual we put $\pi(t;4,3) := \sum_{p\le t,\, p\equiv 3\,(\mathrm{mod}\,4)} 1$. By the Siegel–Walfisz theorem

$$\pi(t;4,3) = \frac12\int_2^t\frac{\mathrm{d}v}{\log v} + O\big(t\,e^{-\sqrt{\log t}}\big)$$

and by partial integration, we easily deduce that

$$\begin{aligned}(4.3)\qquad Y &= \frac{1+o(1)}{2}\sum_{n\le x^{1-2/\alpha}}\frac{u(n)f(n)}{n}\int_{x^{1/\alpha}}^{(x/n)^{1/2}}\frac{\mathrm{d}\pi(t;4,3)}{t\log^2(x/nt)}\\ &= \frac{1+o(1)}{4}\sum_{n\le x^{1-2/\alpha}}\frac{u(n)f(n)}{n}\int_{x^{1/\alpha}}^{(x/n)^{1/2}}\frac{\mathrm{d}t}{\log^2(x/nt)\,t\log t}\\ &= \frac{1+o(1)}{4\log^2 x}\sum_{n\le x^{1-2/\alpha}}\frac{u(n)f(n)}{nh(n)^2}\Big(\frac{\alpha h(n)-2}{\alpha h(n)-1}+\log\big(\alpha h(n)-1\big)\Big)\end{aligned}$$

with $h(n) := 1 - \log n/\log x$. We define

$$U(t) := \sum_{n\le t} u(n)f(n), \qquad K(t) := \frac{1}{th(t)^2}\Big(\frac{\alpha h(t)-2}{\alpha h(t)-1}+\log\big(\alpha h(t)-1\big)\Big).$$

It is not difficult to show that we have, uniformly for $x\ge 10$ and $1\le t\le x^{1-2/\alpha}$,

$$(4.4)\qquad K'(t) = -\frac{1}{t^2h(t)^2}\Big(\frac{\alpha h(t)-2}{\alpha h(t)-1}+\log\big(\alpha h(t)-1\big)\Big)+O\Big(\frac{1}{t^2\log x}\Big).$$

Let Z be the last sum on the right-hand side of (4.3). Noticing that $U(1-) = K(x^{1-2/\alpha}) = 0$, using (4.1) and (4.4), we deduce, by partial integration, that

$$\begin{aligned}Z &= \int_{1-}^{x^{1-2/\alpha}} K(v)\,\mathrm{d}U(v) = -\int_{1}^{x^{1-2/\alpha}} U(v)K'(v)\,\mathrm{d}v\\ &= \frac{A}{C_1}\int_1^{x^{1-2/\alpha}}\Big(\frac{\alpha h(v)-2}{\alpha h(v)-1}+\log\big(\alpha h(v)-1\big)\Big)\frac{\mathrm{d}v}{vh(v)^2(\log v)^{1/2}}+O(1).\end{aligned}$$

By the change of variables $t = \alpha h(v)$, we obtain

$$Z = \sqrt{\log x}\,\frac{A\alpha}{C_1}\int_2^{\alpha}\frac{t-2+(t-1)\log(t-1)}{t^2(t-1)(1-t/\alpha)^{1/2}}\,\mathrm{d}t+O(1).$$

Inserting this in (4.3) yields (4.2). This completes the proof. □

An upper bound for T is obtained in the following proposition.

Proposition 2. *Assume that* $\frac{4}{5}<\theta<1$ *and* $2\le\alpha<\min\{3,\,2/(5-5\theta)\}$. *We then have*

$$T \le \{W_2(\theta,\alpha)+o(1)\}\,x^{\theta}/(\log x)^{3/2},$$

where

$$W_2(\theta,\alpha) := \frac{AC_3\alpha}{2\theta-1}\int_2^{\alpha}\frac{t-2+(t-1)\log(t-1)}{t^2(t-1)(1-t/\alpha)^{1/2}}\,\mathrm{d}t,$$

and A, C_3 *are defined as in Proposition 1.*

Proof. For every $l\in\mathcal{L}$, it is natural to choose, in Lemma 1,

$$\mathcal{F} = \mathcal{M}(l), \quad \mathcal{P} = \mathcal{P}(l), \quad X = \frac{1}{2}\int_{x/l}^{(x+x^{\theta})/l}\frac{\mathrm{d}t}{\log t},$$

$$w(p) = \begin{cases} p/(p-1) & \text{if } p\in\mathcal{P}(l),\\ 0 & \text{otherwise.}\end{cases}$$

Let d be a squarefree integer whose prime factors belong to $\mathcal{P}(l)$. By the Chinese Remainder Theorem, the system of congruences $(l/2)p_1 \equiv 1 \pmod 4, lp_1 \equiv -1 \pmod d$ has exactly one solution $a^* \pmod{4d}$. Thus we have

$$|\mathcal{M}(l)_d| = \sum_{\substack{x/l<p_1\le(x+x^\theta)/l\\ p_1\equiv a^* \pmod{4d}}} 1, \qquad r(\mathcal{M}(l),d) = H(x/l, x^\theta/l, 4d, a^*, 1).$$

According to Lemma 1, one has

(4.5)
$$S\big(\mathcal{M}(l);\mathcal{P}(l),(x/l)^{\theta_1}\big) \le XV\big((x/l)^{\theta_1}\big)\Big\{F\Big(\frac{\log Q}{\theta_1\log(x/l)}\Big)+o(1)\Big\} + R(x,x^\theta,Q,l),$$

where $\theta_1 = (2\theta-1)/4$ and

$$R(x,x^\theta,Q,l) := \sum_{d<Q,\, d|P((x/l)^{\theta_1})} 3^{\omega(d)}|r(\mathcal{M}(l),d)|.$$

Using Mertens' formula, we have

$$V\big((x/l)^{\theta_1}\big) = \prod_{p<(x/l)^{\theta_1},\,(p,l)=1}\Big(1-\frac{1}{p-1}\Big) = \{1+o(1)\}\frac{2C_1C_3e^{-\gamma}f(l)}{\theta_1\log(x/l)}, \tag{4.6}$$

where $f(l)$, C_1 are defined as in Lemma 3. Since $\frac{4}{5}<\theta<1$ and $2\le\alpha<2/(5-5\theta)$, we have $\max_{l\in\mathcal{L}} l = 2x^{1-1/\alpha} \le x^{(5\theta-3)/2-\varepsilon}$ and $x^\theta/l \ge (x/l)^{3/5+\varepsilon/5}$. Taking $Q = (x/l)^{\theta-1/2}/\log^B(x/l)$ and $g(1)=1$, $g(m)=0$ $(m\ge 2)$, it follows from (2.2) of Lemma 2 that

$$R(x,x^\theta,Q,l) \ll x^\theta/l\log^3(x/l). \tag{4.7}$$

Combining (4.5)–(4.7), one has the following estimate

$$S\big(\mathcal{M}(l);\mathcal{P}(l),(x/l)^{\theta_1}\big) \le \{1+o(1)\}\frac{C_1C_3f(l)x^\theta}{\theta_1 l\log^2(x/l)}.$$

Then summing over l and using Lemma 4, we obtain the desired inequality. □

§5. THE END OF THE PROOF OF THE THEOREM

Assuming that

$$\tfrac{6}{7}<\theta<1 \quad\text{and}\quad 2/(2\theta-1)\le\alpha<\min\{3, 2/(5-5\theta)\}, \tag{5.1}$$

we have, by (1.3)–(1.6) and Propositions 1–2, that

$$\sum_{x<p\le x+x^\theta} b^*(p-1) \ge \big\{\big(AC_3/(2\theta-1)\big)W(\theta,\alpha)+o(1)\big\}\,x^\theta/(\log x)^{3/2},$$

where

(5.2)
$$W(\theta,\alpha) := \sqrt{\theta-1/2}\int_1^{\alpha(\theta-1/2)}\frac{\mathrm{d}t}{\sqrt{t(t-1)}} - \alpha\int_2^\alpha \frac{t-2+(t-1)\log(t-1)}{t^2(t-1)(1-t/\alpha)^{1/2}}\,\mathrm{d}t.$$

In order to facilitate the calculation on a computer, we eliminate, by integration by parts, the singularity of two integrands on the right-hand side of (5.2). We have

$$\begin{aligned} W(\theta,\alpha) =& \sqrt{4\theta-2-4/\alpha}+\sqrt{\theta-1/2}\int_1^{\alpha(\theta-1/2)}(t-1)^{1/2}t^{-3/2}\,\mathrm{d}t \\ &-2\alpha^2\int_2^{\alpha}\frac{-t^2+6t-4-2(t-1)^2\log(t-1)}{(t-1)^2t^3}(1-t/\alpha)^{1/2}\,\mathrm{d}t. \end{aligned}$$

We choose $\theta=\frac{115}{121}$ and $\alpha=2.349$, which satisfy (5.1). A numerical computation gives us

$$\begin{aligned} &W\big(\tfrac{115}{121},2.349\big) \\ &\quad =0.314325\cdots+0.671128\cdots\times 0.008853\cdots-2\times 2.349^2\times 0.028982\cdots \\ &\quad \ge 4\times 10^{-4}. \end{aligned}$$

This completes the proof.

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