

## EXTENDING THE FORMULA TO CALCULATE THE SPECTRAL RADIUS OF AN OPERATOR

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ABSTRACT. In a Banach space, Gelfand's formula is used to find the spectral radius of a continuous linear operator. In this paper, we show another way to find the spectral radius of a bounded linear operator in a complete topological linear space. We also show that Gelfand's formula holds in a more general setting if we generalize the definition of the norm for a bounded linear operator.

### 1. INTRODUCTION AND BASIC DEFINITIONS

In all that follows  $E$  stands for a linear vector space over the field  $\mathbf{C}$  of complex numbers.  $E[t]$  will denote a complete locally convex topological vector space, with a Hausdorff topology  $t$ , and  $T : E \rightarrow E$  will be a linear map. Finally,  $\vartheta(t)$  will be the filter of all balanced, convex and closed  $t$ -neighborhoods of zero (in  $E$ ).

**Definition 1.** The linear operator  $T : E[t] \rightarrow E[t]$  is said to be a bounded operator, if there is a neighborhood  $U \in \vartheta(t)$  such that  $T(U)$  is a bounded set.

If in the definition above  $T(U)$  is a relatively compact set, then  $T$  is said to be a compact operator. Any compact operator is a bounded operator, and any bounded operator is continuous (with the  $t$ -topology) (see [5]).

We recall that, given any topological linear space  $E[\omega]$  and  $S : E[\omega] \rightarrow E[\omega]$  a linear operator, the resolvent of  $S$  is the set

$$\rho_\omega(S) = \left\{ \xi \in \mathbf{C} \mid \xi I - S : E[\omega] \rightarrow E[\omega] \right. \\ \left. \text{is bijective and has a continuous inverse} \right\}.$$

The spectrum of  $S$  is defined by  $\sigma_\omega(S) = \mathbf{C} \setminus \rho_\omega(S)$  (the set-theoretic complement in  $\mathbf{C}$  of the resolvent set), and the spectral radius by

$$sr_\omega(S) = \sup \left\{ |\lambda| \mid \lambda \in \sigma_\omega(S) \right\}.$$

**Definition 2.** A net  $\{x_\alpha\}_J \subset E$  is said to be  $t$ -ultimately bounded ( $t$ -ub) if, given any  $V \in \vartheta(t)$ , there is a positive real number  $r$  and an index  $\alpha_0 \in J$ , both depending on  $V$ , such that  $x_\alpha \in rV \forall \alpha \geq \alpha_0$ .

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Let us denote by  $\Gamma$  the set of all  $t$ -ub nets in  $E$ .

*Remark 1.* Any bounded, convergent or Cauchy net is a  $t$ -ub net. For more details about  $t$ -ub nets we refer the reader to [1].

**Definition 3.** Let  $\xi \in \mathbf{C}$ . We will say that  $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$  ( $T^n = T \circ T \circ \dots \circ T$ ,  $n$  times) if, given both  $V \in \vartheta(t)$  and  $\{x_\alpha\}_J \in \Gamma$ , there exist  $\alpha_0 \in J$  and  $n_0 \in \mathbf{N}$  such that  $\frac{1}{\xi^n}T^n(x_\alpha) \in V \ \forall \alpha \geq \alpha_0$  and  $\forall n \geq n_0$ .

**Definition 4.**  $\gamma_t(T) = \inf \left\{ |\xi| \mid \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0 \right\}$ .

*Remark 2.* It is shown by Vera [4] that for a bounded operator  $T$ , we have:

- (i)  $\gamma_t(T) < \infty$ , and for any  $\xi \in \mathbf{C}$  such that  $\gamma_t(T) < |\xi|$ ,  $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$ .
- (ii)  $sr_t(T) \leq \gamma_t(T)$ , where  $sr_t(T)$  is the spectral radius of  $T$ .
- (iii) When  $E[t]$  is a Banach space,  $\gamma_t(T) = r_t(T)$ .

In [2] it was proved, based in the above result, that  $\gamma_t(T) = sr_t(T)$  when  $T$  is a compact operator. In this paper we extend that result to any bounded operator.

## 2. MAIN RESULTS

From now on let  $T : E[t] \rightarrow E[t]$  be a bounded operator and let  $U \in \vartheta(t)$  be such that  $T(U)$  is a bounded set.

Let  $P_U$  be the functional of Minkowski (see [3]) generated by  $U$ , which is a seminorm on  $E$ . Let  $E[P_U]$  denote the vector space  $E$  with the topology given by the seminorm  $P_U$ .

*Remark 3.* The topology on  $E$  given by the seminorm  $P_U$  is coarser than the topology  $t$  ( $P_U \leq t$ ).

**Proposition 1.**  $T : E[P_U] \rightarrow E[P_U]$  is a bounded operator (hence a continuous one).

*Proof.* Since  $T(U)$  is a bounded set and  $P_U \leq t$ ,  $T(U)$  is also a  $P_U$ -bounded set in  $E[P_U]$ .  $\square$

**Definition 5.**  $\gamma_{P_U}(T) = \inf \left\{ |\xi| \mid \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{P_U}} 0 \right\}$ .

Here  $\Gamma_{P_U}$  convergence means that, given any net  $\{x_\alpha\}_J \subset E$  such that for all  $\alpha$ ,  $P_U(x_\alpha) \leq r$  for some positive real number  $r$  ( $P_U$ -bounded net), then  $P_U(\frac{1}{\xi^n}T^n x_\alpha) \rightarrow 0$  as a net in  $\mathbf{R}$  whose index set is  $\mathbf{N} \times \mathbf{J}$ .

**Proposition 2.**  $\gamma_{P_U}(T) = \gamma_t(T)$ .

*Proof.* Let  $\xi \in \mathbf{C}$  be such that  $\gamma_{P_U}(T) < |\xi|$ , and let  $V \in \vartheta(t)$  and  $\{x_\alpha\}_J \in \Gamma$  be given. Since  $\frac{1}{\xi}T(U)$  is a bounded set, there is a positive real number  $r_1$  such that  $\frac{1}{r_1\xi}T(U) \subset V$ . In [1] is shown that  $\{x_\alpha\}_J \in \Gamma \Rightarrow \{r_1x_\alpha\}_J \in \Gamma$ . This implies that there exist both  $\alpha_0 \in J$  and  $r_2 > 0$  such that  $r_1x_\alpha \in r_2U \ \forall \alpha \geq \alpha_0$ , i.e.,  $P_U(r_1x_\alpha) \leq r_2$ , that is, the net  $\{x_\alpha\}_{\alpha \geq \alpha_0}$  is a  $P_U$ -bounded net; therefore,  $\exists \alpha_1 \in J$  ( $\alpha_1 \geq \alpha_0$ ) and  $n_1 \in \mathbf{N}$  such that  $P_U(\frac{1}{\xi^n}T^n(x_\alpha)) < 1 \ \forall \alpha \geq \alpha_1, n \geq n_1$ , that is,  $\frac{1}{\xi^n}T^n(x_\alpha) \in U$  for those indices. Hence

$$\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha = \frac{1}{r_1\xi}T\left(\frac{1}{\xi^n}T^n r_1x_\alpha\right) \in \frac{1}{r_1\xi}T(U) \subset V \ \forall \alpha \geq \alpha_1, n \geq n_1,$$

that is,  $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$ , and therefore,  $\gamma_t(T) \leq |\xi|$ . This implies that  $\gamma_{P_U}(T) \leq \gamma_t(T)$ .

On the other hand, let  $\gamma_t(T) < |\xi|$  and  $\{x_\alpha\}_J$ , a  $P_U$ -bounded net; that is,  $x_\alpha \in rU$  for all  $\alpha$  and some  $r > 0$ . Then  $\{\frac{1}{\xi}Tx_\alpha\}_J \subset \frac{r}{\xi}T(U)$ , where  $\frac{r}{\xi}T(U)$  is a  $t$ -bounded set; therefore,  $\{\frac{1}{\xi}Tx_\alpha\}_J \in \Gamma$ . Since  $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$ , given  $\epsilon > 0$ ,  $\exists \alpha_0 \in J$  and  $n_0 \in \mathbf{N}$  such that  $\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha = \frac{1}{\xi^n}T^n(\frac{1}{\xi}Tx_\alpha) \in \epsilon U$   $\forall \alpha \geq \alpha_0, n \geq n_0$ ; that is,  $P_U(\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha) \leq \epsilon$  for those indices. This says that  $\frac{1}{\xi^n}T^n x_\alpha$  is  $P_U$ -convergent to 0; therefore,  $\gamma_{P_U}(T) \leq |\xi|$ . This implies that  $\gamma_t(T) \leq \gamma_{P_U}(T)$ .  $\square$

**Definition 6.**

$$L(E) = \left\{ S : E[t] \rightarrow E[t] \mid S \text{ is a linear and continuous operator} \right\},$$

$$L_U(E) = \left\{ S \in L(E) \mid S(U) \text{ is a bounded set} \right\},$$

$L_U(E)$  is a vector subspace of the complex vector space  $L(E)$ .

*Remark 4.* For the bounded operator  $T$  that we have been working on we have  $T, T^n, \lambda T, \lambda T^n \in L_U(E)$  for all  $n \in \mathbf{N}$  and all  $\lambda \in \mathbf{C}$ .

Moreover, for any  $S \in L(E)$ ,  $S \circ T, T \circ S \in L_U(E)$ .

**Definition 7.** For any operator  $S \in L_U(E)$ , we define, taking into account that  $S(U)$  is a bounded set, the following real number:

$$\|S\|_U = \sup\{P_U(Sx) \mid x \in U\}.$$

It easy to check that  $\|S^n\|_U \leq \|S\|_U^n \forall S \in L_U(E)$  and  $\forall n \in \mathbf{N}$ .

**Theorem 1.** If  $S_n \xrightarrow{\Gamma_t} S$  in  $L(E)$ , then  $\|S_n \circ T - S \circ T\|_U \rightarrow 0$ .

*Proof.* Let us prove it by way of contradiction.

Let  $\epsilon > 0$  be such that there exist natural numbers  $n_1 < n_2 < n_3 < \dots$  such that  $\epsilon < \|S_{n_k} \circ T - S \circ T\|_U$ ; hence, for each of those  $n_k$  there is  $x_{n_k} \in U$  such that  $P_U[(S_{n_k} \circ T - S \circ T)x_{n_k}] > \epsilon$ . Since  $\{Tx_{n_k}\} \subset T(U)$ , it is a bounded sequence; hence, for  $V = \epsilon U \in \vartheta(t)$  there is an index  $m_0 \in \mathbf{N}$  such that  $(S_n - S)(Tx_{n_k}) \in V$  for all  $n, n_k \geq m_0$ ; this implies that  $P_U[(S_{n_k} \circ T - S \circ T)x_{n_k}] \leq \epsilon$ , which yields a contradiction.  $\square$

**Proposition 3.**  $\rho_t(T) \subset \rho_{P_U}(T)$ .

*Proof.* Let us suppose first that  $\gamma_t(T) < 1$ . Let  $\xi \in \rho_t(T)$  be such that  $|\xi| > \gamma_t(T)$ . Then  $S = \sum_{k=0}^{\infty} \frac{1}{\xi^{k+1}}T^k$  is a continuous operator and  $S = (\xi I - T)^{-1}$ . Set  $S_n = \sum_{k=0}^n \frac{1}{\xi^{k+1}}T^k$ . Then  $S_n \xrightarrow{\Gamma_t} S$ , and from Theorem 1 it follows that  $\|S_n \circ \frac{1}{\xi}T - S \circ \frac{1}{\xi}T\|_U \rightarrow 0$ . On the other hand,  $S_n \circ \frac{1}{\xi}T = S_{n+1} - \frac{1}{\xi}I$  and  $S \circ \frac{1}{\xi}T = S - \frac{1}{\xi}I$ ; hence  $\|S_{n+1} - S\|_U \rightarrow 0$ . Thereby, given  $\{x_m\}_{\mathbf{N}} \subset E$  such that  $P_U(x_m) \rightarrow 0$ , then  $P_U(Sx_m) \leq P_U[(S - S_n)x_m] + P_U(S_n x_m) \rightarrow 0$ . This proves that  $S : E[P_U] \rightarrow E[P_U]$  is a continuous operator; hence  $\xi \in \rho_{P_U}(T)$ .

Now let  $\xi \in \rho_t(T)$  be such that  $|\xi| \leq \gamma_t(T)$ . Then  $|\frac{1}{\xi}| > 1 > \gamma_t(T)$ , which means that  $\frac{1}{\xi}I - T : E[P_U] \rightarrow E[P_U]$  is a continuous operator. Since  $\xi I - T =$

$(\xi - \frac{1}{\xi})I - (T - \frac{1}{\xi}I)$ , we have that

$$(\xi I - T)^{-1} = (\xi - \frac{1}{\xi})^{-1}I \circ [(T - \frac{1}{\xi}I)^{-1} - (\frac{1}{\xi})^{-1}I] \circ (T - \frac{1}{\xi}I)^{-1};$$

since the right hand side is the composition of three continuous operators from  $E[P_U]$  to  $E[P_U]$  we have that  $\xi \in \rho_{P_U}(T)$ .

Finally, let  $T$  be such that  $\gamma_t(T) < r < \infty$ . Then  $T_1 = \frac{1}{r}T \in L_U(E)$  is such that  $\gamma_t(T_1) < 1$ . Hence  $\frac{1}{r}\rho_t(T) = \rho_t(T_1) \subset \rho_{P_U}(T_1) = \frac{1}{r}\rho_{P_U}(T)$ , and therefore  $\rho_t(T) \subset \rho_{P_U}(T)$ .  $\square$

**Definition 8.**  $N = \left\{ x \in E \mid P_U(x) = 0 \right\}$ .

*Remark 5.* Since  $\left\{ x \in E \mid P_U(x) \leq 1 \right\} \subset U$ ,  $N \subset U$ .

**Theorem 2.**  $N$  is a closed linear subspace of  $E$ , and  $T(x) = 0$  for all  $x \in N$ .

*Proof.* The first claim follows from the fact that

$$P_U(\xi x + y) \leq |\xi| P_U(x) + P_U(y).$$

For the second claim let's take  $x \in N$ ; then  $mx \in N$  for  $m = 1, 2, \dots$ . Let  $V$  be any balanced, convex and closed  $t$ -neighborhood of 0. Since  $\{mT(x)\}_{m=1,2,3,\dots} \subset T(N) \subset T(U)$  and the latter set is bounded, there exists  $r \in \mathbf{R}^+$  such that  $\{mT(x)\} \subset rV \Rightarrow T(x) \in \frac{r}{m}V \subset V$  when  $m > r$ . Since  $V$  was an arbitrary neighborhood of zero and  $E[t]$  is Hausdorff, then  $T(x) = 0$ .  $\square$

**Definition 9.** Let  $E/N$  be the quotient linear space and let  $\hat{P}_U$  be the norm on it defined by  $\hat{P}_U(x + N) = P_U(x)$  (see [3]).

*Remark 6.*  $(E/N)[\hat{P}_U]$  will denote the vector space  $E/N$  with the topology given by the norm  $\hat{P}_U$ .

**Definition 10.** Let  $\hat{T} : (E/N) \rightarrow (E/N)$  be defined by  $\hat{T}(x + N) = T(x) + N$ .

*Remark 7.* It is easy to show that  $\hat{T}$  is a well defined linear map.

**Proposition 4.**  $\hat{T} : (E/N)[\hat{P}_U] \rightarrow (E/N)[\hat{P}_U]$  is a linear and bounded operator (hence  $\hat{T}$  is continuous).

*Proof.*  $U/N$  is the unit ball in  $(E/N)[\hat{P}_U]$  and  $\hat{T}(U/N) = (T(U) + N)/N$ . The latter set is  $\hat{P}_U$ -bounded because the canonical projection  $E[P_U] \rightarrow (E/N)[\hat{P}_U]$  is a continuous map.  $\square$

*Remark 8.* Since  $(E/N)[\hat{P}_U]$  is a norm space we can define, as usual, the norm of  $\hat{T}$ , and this will be denoted by  $\|\hat{T}\|_{\hat{P}_U}$ .

**Proposition 5.**  $\gamma_{\hat{P}_U}(\hat{T}) = \gamma_{P_U}(T)$ .

*Proof.* Set  $|\xi| > \gamma_{P_U}(T)$ . Let  $\{x_\alpha + N\}_J$  be a  $\hat{P}_U$ -bounded net in  $E/N$ ; then  $\{x_\alpha\}_J$  is a  $P_U$ -bounded net in  $E$ ; hence, given  $\epsilon > 0$ , there are indices  $\alpha_0 \in J$  and  $n_0 \in \mathbf{N}$  such that  $\frac{1}{\xi^n}T^n x_\alpha \in \epsilon U \forall \alpha \geq \alpha_0$  and  $n \geq n_0$ . Thus

$$\frac{1}{\xi^n}\hat{T}^n(x_\alpha + N) = \frac{1}{\xi^n}T^n x_\alpha + N \in \epsilon(U/N), \quad \alpha \geq \alpha_0, \quad n \geq n_0$$

This implies that  $\gamma_{\hat{P}_U}(\hat{T}) \leq |\xi|$ . Hence  $\gamma_{\hat{P}_U}(\hat{T}) \leq \gamma_{P_U}(T)$ .

Set  $|\xi| > \gamma_{\hat{P}_U}(T)$ . Let  $\{x_\alpha\}_J$  be a  $P_U$ -bounded net in  $E$ . Then  $\{x_\alpha + N\}_J$  is a  $\hat{P}_U$ -bounded net in  $E/N$ ; hence, given  $\epsilon > 0$ , there are indices  $\alpha_0 \in J$  and  $n_0 \in \mathbf{N}$  such that  $\frac{1}{\xi^n} \hat{T}^n(x_\alpha + N) \in \epsilon(U/N) \forall \alpha \geq \alpha_0, n \geq n_0$ . This implies that for those indices  $\frac{1}{\xi^n} T^n x_\alpha = \epsilon u_\alpha + z_\alpha, u_\alpha \in U, z_\alpha \in N$ ; hence  $P_U(\frac{1}{\xi^n} T^n x_\alpha) \leq P_U(\epsilon u_\alpha) + P_U(z_\alpha) \leq \epsilon + 0 = \epsilon$ , and thus  $|\xi| > \gamma_{P_U}(T)$ . This implies that  $\gamma_{\hat{P}_U}(T) \geq \gamma_{P_U}(T)$ .  $\square$

**Proposition 6.**  $\rho_{P_U}(T) = \rho_{\hat{P}_U}(\hat{T})$ .

*Proof.*  $\xi \in \rho_{P_U}(T) \Rightarrow \xi I - T : E[P_U] \rightarrow E[P_U]$  is bijective and has a continuous inverse.

Let us show that  $A : (E/N)[\hat{P}_U] \rightarrow (E/N)[\hat{P}_U]$  defined by  $A(x + N) = (\xi I - T)^{-1}(x) + N$ , which is a linear and continuous map, is the inverse function of  $\xi \hat{I} - \hat{T}$ . For this,  $A(\xi \hat{I} - \hat{T})(x + N) = A(\xi \widehat{I - T})(x + N) = A((\xi I - T)(x) + N) = (\xi I - T)^{-1}(\xi I - T)(x) + N = x + N$ . In a similar way it can be proved that  $(\xi \hat{I} - \hat{T}) \circ A = I$ . This implies that  $\xi \in \rho_{\hat{P}_U}(\hat{T})$ .

It is just routine to prove the set contention in the other way around.  $\square$

**Definition 11.**  $(\widetilde{E/N})[\tilde{P}_U]$  will denote the completion (as a normed space) of  $(E/N)[\hat{P}_U]$ , and  $\tilde{T}$  will denote the natural extension of  $\hat{T}$ .

*Remark 9.*  $(\widetilde{E/N})[\tilde{P}_U]$  is a Banach space. Besides, since  $\hat{T}$  is a bounded operator,  $\tilde{T} : (\widetilde{E/N})[\tilde{P}_U] \rightarrow (\widetilde{E/N})[\tilde{P}_U]$  is a bounded operator (see [3]).

*Remark 10.* Since  $(\widetilde{E/N})[\tilde{P}_U]$  is a Banach space we can define, as usual, the norm of  $\tilde{T}$ ; this will be denoted by  $\|\tilde{T}\|_{\tilde{P}_U}$ .

**Proposition 7.**  $\gamma_{\hat{P}_U}(\hat{T}) = \gamma_{\tilde{P}_U}(\tilde{T})$ .

*Proof.* Since  $\tilde{T}$  is an extension of  $\hat{T}$ , the proof follows immediately from the definitions of  $\gamma_{\hat{P}_U}(\hat{T})$  and  $\gamma_{\tilde{P}_U}(\tilde{T})$ .  $\square$

**Proposition 8.**  $\rho_{\tilde{P}_U}(\tilde{T}) = \rho_{\hat{P}_U}(\hat{T})$ .

*Proof.* If  $\xi \in \rho(\hat{T})$ , then  $\xi I - \hat{T} : (E/N)[\hat{P}_U] \rightarrow (E/N)[\hat{P}_U]$  is bijective and has a continuous inverse, so that both  $\xi I - \hat{T}$  and  $(\xi I - \hat{T})^{-1}$  have a continuous extension to  $(\widetilde{E/N})$ , which are precisely  $\xi I - \tilde{T}$  and  $(\xi I - \tilde{T})^{-1}$  respectively. This implies that  $\xi \in \rho(\tilde{T})$ .

On the other hand, if  $\xi \in \rho(\tilde{T})$ , then  $\xi I - \tilde{T} : (\widetilde{E/N})[\tilde{P}_U] \rightarrow (\widetilde{E/N})[\tilde{P}_U]$  is bijective and has a continuous inverse; hence the restrictions of those functions to  $(E/N)[\hat{P}_U]$  are precisely  $\xi I - \hat{T}$  and its inverse function, which are continuous functions for being the restrictions of continuous ones. Then  $\xi \in \rho(\hat{T})$ .  $\square$

**Theorem 3.**  $\gamma_t(T) = sr_t(T)$ .

*Proof.* By Remark 2 (ii) it suffices to show that  $sr_t(T) \geq \gamma_t(T)$ . Also, from Remark 2 (iii) we get

$$(1) \quad sr_{\tilde{P}_U}(\tilde{T}) = \gamma_{\tilde{P}_U}(\tilde{T})$$

because  $\widetilde{(E/N)[\tilde{P}_U]}$  is a Banach space. From Propositions 2, 5 and 7 we obtain

$$(2) \quad \gamma_t(T) = \gamma_{\tilde{P}_U}(\tilde{T})$$

From Propositions 3, 6 and 8 we obtain

$$\rho_t(T) \subset \rho_{\tilde{P}_U}(\tilde{T});$$

this implies that

$$(3) \quad sr_{\tilde{P}_U}(\tilde{T}) \leq sr_t(T).$$

From (1), (2) and (3) we finally get

$$\gamma_t(T) \leq sr_t(T).$$

□

### 3. A GENERALIZATION OF GELFAND'S FORMULA

In this part we prove that Gelfand's formula (see [3]) applies for a bounded operator defined on a topological vector space. Following the notation from the sections above, we will show that we can use  $\|T\|_U$  in Gelfand's formula to calculate the spectral radius of  $T$ .

**Proposition 9.** *For any  $T \in L_U(E)$ ,  $\|T\|_U = \|\hat{T}\|_{\hat{P}_U}$ .*

*Proof.* Set  $r > \|T\|_U$ . Then  $T(U) \subset rU$ ; hence  $P_U(Tx) \leq r$  for all  $x \in U$ . This implies that  $\|\hat{T}\|_{\hat{P}_U} \leq r$ , and therefore  $\|\hat{T}\|_{\hat{P}_U} \leq \|T\|_U$ .

Set  $r < \|T\|_U$ . Then there exists  $x \in U$  such that

$$r < P_U(Tx) = \hat{P}_U(\hat{T}(x + N)) \leq \|\hat{T}\|_{\hat{P}_U}.$$

This implies that  $\|\hat{T}\|_{\hat{P}_U} = \|T\|_U$ . □

**Corollary 1.**  $\|\tilde{T}\|_{\tilde{P}_U} = \|\hat{T}\|_{\hat{P}_U} = \|T\|_U$ .

**Theorem 4.**  $sr_t(T) = \lim_{n \rightarrow \infty} \|T^n\|_U^{\frac{1}{n}}$  for any  $T \in L_U(E)$ .

*Proof.* We recall first that  $T \in L_U(E) \Rightarrow T^n \in L_U(E)$ . From (1), (2), and Theorem 3 we obtain

$$(4) \quad sr_t(T) = sr_{\tilde{P}_U}(\tilde{T}).$$

Because  $\widetilde{(E/N)[\tilde{P}_U]}$  is a Banach space, Gelfand's formula holds:

$$(5) \quad sr_{\tilde{P}_U}(\tilde{T}) = \lim_{n \rightarrow \infty} \|\tilde{T}^n\|_{\tilde{P}_U}^{\frac{1}{n}}.$$

Finally, using (4) and (5) and the corollary above, we obtain

$$sr_t(T) = \lim_{n \rightarrow \infty} \|T^n\|_U^{\frac{1}{n}}.$$

□

## REFERENCES

- [1] C. L. DeVito, *On Alaoglu's Theorem, Bornological Spaces and the Mackey-Ulam Theorem*. Math. Ann. **192**, 83-89 (1972). MR **44**:2014
- [2] F. Garibay and R. Vera, *A Formula to Calculate the Spectral Radius of a Compact Linear Operator*. Internat J. Math. & Math. Sci. **20**, no. 3 (1997), 585-588.
- [3] W. Rudin, *Functional Analysis*. New York, McGraw-Hill, Inc. 1991. MR **92k**:46001
- [4] R. Vera, *Linear Operators on Locally Convex Topological Vector Spaces*. Ph.D. Thesis, Department of Mathematics, University of Arizona, 1994.
- [5] K. Yosida, *Functional Analysis*. Springer Verlag, New York, 1965. MR **31**:5054

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