

ON CONFORMAL DISTORTION AND SULLIVAN'S SECTOR THEOREM

EDSON DE FARIA

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ABSTRACT. Using general bounds on the conformal distortion of univalent maps, we prove a strong version of Sullivan's sector theorem, which gives certain sufficient conditions for an arbitrarily long composition of univalent Herglotz functions to map the upper half-plane into a proper sub-sector.

1. INTRODUCTION

The purpose of this article is to generalize the *sector theorem* of D. Sullivan. A simplified statement of this theorem says that a long composition of *square-root-like* mappings in the upper half-plane \mathbb{C}^+ will map \mathbb{C}^+ into a smaller sector of angle $< \pi$ independent of the length of the composition, provided their *poles* diverge towards $\pm\infty$ along the real axis at an exponential rate. The sector theorem was used by Sullivan in [8] to establish complex a-priori bounds for renormalization of quadratic-like mappings (*cf.* also [7]). The robust estimates given by this powerful tool can be used to prove strong compactness properties of certain composition operators acting on analytic mappings, such as the renormalization operators appearing in [8]. The more general version presented here was used by the author in [3] to prove similar results for critical homeomorphisms of the circle.

We proceed to a precise statement of the generalization we have in mind. Given $a, b \in \mathbb{R}$ with $a < b$, let $\mathcal{H}_{a,b}$ be the class of all univalent mappings ϕ defined on $\mathbb{C}(I_\phi) = (\mathbb{C} \setminus \mathbb{R}) \cup I_\phi$, where $I_\phi \supseteq (a, b)$ is some open interval, which preserve both half-planes $\mathbb{C}^+, \mathbb{C}^-$ and are such that $\phi((a, b)) = (a, b)$. Members of this class are normalized, univalent *Herglotz functions* (*cf.* [4], [9]). We refer to I_ϕ as the *base* of $\phi \in \mathcal{H}_{a,b}$: it is the largest interval containing (a, b) restricted to which ϕ is a homeomorphism into the reals. An element $A \in \mathcal{H}_{a,b}$ is a *left α -root* ($0 < \alpha < 1$) if there exists $a_0 \leq a$ such that $A(z) = u(z - a_0)^\alpha + v$, where $u, v \in \mathbb{R}$ and the branch of $z \mapsto (z - a_0)^\alpha$ is uniquely determined by the requirements $A(a) = a$, $A(b) = b$. The point $a_0 \in \mathbb{R}$ is the *pole* of A . Right roots can be similarly defined, but will not be used here. Given a bounded interval $J \subseteq \mathbb{R}$ and any $\lambda > 0$, we denote by J^λ the closed interval centered at the midpoint of J whose length is $(1 + \lambda)$ -times the length of J .

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Theorem 1 (Generalized sector theorem). *Let there be given $A_i, B_i \in \mathcal{H}_{a,b}$, for $i = 1, 2, \dots, m$, and constants $\lambda, K, s > 0$ and $0 < \alpha < 1$ satisfying the following conditions:*

- (a) *Each A_i is a left α_i -root with $\alpha_i \leq \alpha$ and pole at a_i , where $a_1 = a$ and $a_i < a$ for all $i \geq 2$.*
- (b) *There exists a finite sequence $1 = i_0 < i_1 < \dots < i_q = m$ with $i_{n+1} - i_n \leq s$ such that, if $d_n = \min\{|a_i - a| : i_n \leq i < i_{n+1}\}$, then $\sum_{j \geq n} d_j^{-1} \leq K d_n^{-1}$ for all n .*
- (c) *For all $i \geq 2$, if I_i is the base of B_i then $B_i(I_i) \supseteq [a_i, b]^\lambda$, and, setting $J_i = B_i^{-1}([a_i, b])$, one has $J_i^\lambda \subseteq I_i$ also.*

Under these assumptions, there exists a positive angle $\theta = \theta(\alpha, s, K, \lambda)$ such that the image of the upper half-plane by the composition $A_m B_m \dots A_i B_i \dots A_1 B_1$ is contained in the sector $0 \leq \arg(z - a) \leq \pi - \theta$.

A comparison with Sullivan's original statement in [8] reveals that the main difference lies in hypothesis (b). In Sullivan's version, it is required that the poles a_i , following their order of occurrence in the composition, be grouped together into scales, each containing a bounded number of poles, and that the scales diverge monotonically towards $-\infty$ at an exponential rate. By contrast, our hypothesis (b) is somewhat more flexible, since it allows for some *mixing* of such scales. We also allow roots with arbitrary angles ($\leq \pi\alpha < \pi$), not just square-roots as in the original version.

While the essential ideas of the proof of Theorem 1 remain the same as in the original theorem (Koebe distortion, Schwarz's Lemma, trapping the orbit of a point by means of Poincaré neighborhoods), the actual details of the argument are much more involved. All further tools required in the proof are developed in §2. The complete proof of Theorem 1 is then given in §3.

2. NON-LINEARITY AND ANGLE DISTORTION

Before proving Theorem 1, we recall some classical geometric notions about conformal mappings, and derive some consequences from them in the form of a few lemmas. Let $\phi : \Omega \rightarrow \mathbb{C}$ be a univalent mapping of a domain $\Omega \subseteq \mathbb{C}$, and let $D \subseteq \Omega$ be closed and convex. We define the *total non-linearity* of ϕ on D to be the quantity

$$(1) \quad N_\phi(D) = \text{diam}(D) \sup_{z \in D} \left| \frac{\phi''(z)}{\phi'(z)} \right|.$$

The notation is taken from [6]. The total non-linearity is a monotone non-decreasing function of D , and vanishes identically when ϕ is linear. The reason for assuming D convex in the definition is the following. Since for $z_1, z_2 \in D$ the line segment $[z_1, z_2]$ is contained in D ,

$$\log \left| \frac{\phi'(z_1)}{\phi'(z_2)} \right| \leq \int_{[z_1, z_2]} \left| \frac{\phi''(z)}{\phi'(z)} \right| |dz| \leq N_\phi(D).$$

Therefore,

$$(2) \quad \sup_{z_1, z_2 \in D} \left| \frac{\phi'(z_1)}{\phi'(z_2)} \right| \leq \exp\{N_\phi(D)\}.$$

In other words, the total non-linearity bounds ratios of derivatives on convex domains.

When bounding the non-linearity, the main tool at hand is *Koebe's distortion lemma*, which states that $|\phi''(z)/\phi'(z)| \leq 4/\text{dist}(z, \partial\Omega)$. We shall need also *Koebe's one-quarter theorem*: If $D \subseteq \Omega$ is a disk with center z , then $\phi(D)$ contains a disk about $\phi(z)$ of diameter $\frac{1}{4}|\phi'(z)|\text{diam}(D)$. Both results are corollaries of the *distortion theorem* of Koebe (cf. [1], [2]). Yet another tool we will use is Schwarz's lemma in the invariant formulation due to Pick: Any holomorphic self-mapping of a hyperbolic Riemann surface X weakly contracts the hyperbolic (or Poincaré) metric of X .

Our first lemma gives a simple estimate on how an arbitrary element of $\mathcal{H}_{a,b}$ distorts angles measured from a . Here and throughout we write $\theta(z) = \pi - \arg(z-a)$ to denote the angle that $z-a$ forms with the semi-axis $\{x \leq a\}$. Note that $|\text{Im } z| = |z-a|\sin\theta(z)$.

Lemma 2. *If $\phi \in \mathcal{H}_{a,b}$ and if D is any closed convex set with $a \in D \subseteq \mathbb{C}(I_\phi)$, then for all $z \in D$ we have $|\sin\theta(\phi(z))| \geq \exp\{-N_\phi(D)\}|\sin\theta(z)|$.*

Proof. As ϕ maps \mathbb{C}^+ into itself, the Schwarz-Pick lemma yields

$$\text{Im } \phi(z) \geq |\phi'(z)|\text{Im } z,$$

for all $z \in \mathbb{C}^+$ (consider the upper half-plane with its Poincaré metric). Therefore, for all $z \in D \cap \mathbb{C}^+$, we have by (2) and the mean value inequality

$$\sin\theta(\phi(z)) = \frac{\text{Im } \phi(z)}{|\phi(z) - a|} \geq \frac{|\phi'(z)||z - a|}{|\phi(z) - a|} \sin\theta(z) \geq \exp\{-N_\phi(D)\} \sin\theta(z),$$

and the proof is obviously the same for $z \in D \cap \mathbb{C}^-$. □

In §3 we use the following property of Herglotz mappings, which we call *Poincaré neighborhood trapping*. Consider $\mathbb{C}((a,b))$ with its Poincaré metric ρ , and let γ denote the segment (a,b) , a geodesic in this metric. By the Schwarz-Pick lemma, each element of $\mathcal{H}_{a,b}$ maps each Poincaré neighborhood $\mathcal{P}(r) = \{z \in \mathbb{C}((a,b)) : \rho(z, \gamma) \leq r\}$ into itself.

Note that $\partial\mathcal{P}(r)$ consists of a circular arc with endpoints a, b in the upper half-plane union its mirror image in the lower half-plane (to see why, map $\mathbb{C}((a,b))$ conformally onto $\mathbb{C} \setminus \mathbb{R}^+$ by a Möbius transformation sending (a,b) onto the positive real axis, and then map $\mathbb{C} \setminus \mathbb{R}^+$ conformally onto the upper half-plane using the map $z \mapsto i\sqrt{z}$). In particular, each $z \in \partial\mathcal{P}(r)$ views γ under the same angle $\omega = \omega(r)$ (see Figure 1).

Lemma 3. *Let \mathcal{P}_z be the smallest of the Poincaré neighborhoods $\mathcal{P}(r)$ containing a given point z in the upper half-plane, and let $\mathcal{P}_z^+ = \mathcal{P}_z \cap \mathbb{C}^+$. Then we have $\text{diam } \mathcal{P}_z^+ = |z - b|/\sin\theta(z)$.*

Proof. Apply the law of sines to the triangle azb . □

Finally, we will need the following specific geometric property of *left roots*.

Lemma 4. *There exists a continuous function $\Theta : (0, 1) \times (0, 1) \rightarrow \mathbb{R}^+$ with the following property. Let $A \in \mathcal{H}_{a,b}$ be a left α -root with pole $a_0 < a$, let $0 < R < |a_0 - a|$ and let W be the Euclidean convex-hull of $\{a\} \cup \{z \in \mathbb{C}^+ : |z - a_0| \leq R\}$. Then $A(W)$ contains the triangle in the upper half-plane with base $[A(a_0), a] \subseteq \mathbb{R}$ and angles $\pi\alpha$ at $A(a_0)$ and $\Theta(\alpha, R/|a_0 - a|)$ at a , respectively.*

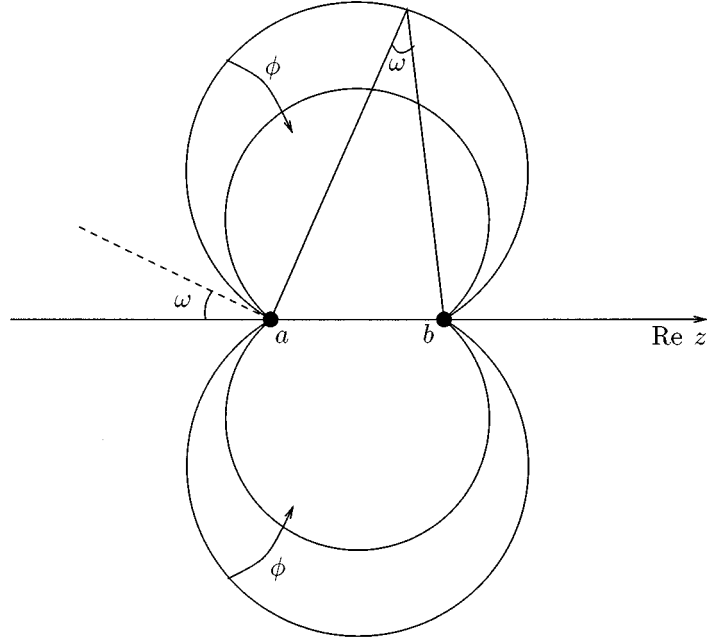


FIGURE 1

Proof. Recall that $A(z) = u(z - a_0)^\alpha + v$, where u, v are determined by the normalization conditions $A(a) = a, A(b) = b$. We first remark that $\theta(Az) > \theta(z)$ for all $z \in \mathbb{C}^+$. To see this, let $L_\varphi = \{a + te^{i\varphi} : t > 0\} \subseteq \mathbb{C}^+$ be a half-line through a , and let H_φ be the component of $\mathbb{C}^+ \setminus L_\varphi$ which lies to the right of L_φ . Computing the curvature of $A(L_\varphi)$, we see that $A(L_\varphi)$ is a convex curve tangent to L_φ at a and $A(L_\varphi) \subseteq H_\varphi$. Therefore $\theta(Az) > \varphi = \theta(z)$ for all $z \in L_\varphi$.

Now, to prove the lemma, it suffices to bound $\theta(Az)$ from below for all $z \in \partial W \cap \mathbb{C}^+$. Let $\omega = \arcsin\{R/|a - a_0|\}$ and note that $\partial W \cap \mathbb{C}^+ = \gamma' \cup \gamma''$, where γ' is the circular arc centered at a_0 with endpoints $x = a_0 - R$ and $y = a_0 + Re^{i(\frac{\pi}{2} - \omega)}$ and γ'' is the line segment joining y to a . Then by the above remark we have $\theta(Az) > \omega$ for all $z \in \gamma''$. Since $A(\gamma')$ is a circular arc centered at $A(a_0)$, to bound $\theta(Az)$ from below for $z \in \gamma'$ it suffices to do so at the endpoints of γ' , and in fact only at x because $y \in \gamma''$ already. Using the equality $v - a = -u(a - a_0)^\alpha$, we get

$$\tan \theta(Ax) = \frac{|uR^\alpha \sin \pi\alpha|}{|uR^\alpha \cos \pi\alpha - u(a - a_0)^\alpha|} = \frac{\sin \pi\alpha}{\left(\frac{a - a_0}{R}\right)^\alpha - \cos \pi\alpha}.$$

Hence, we can define $\Theta(\cdot, \cdot)$ by the formula

$$\Theta(\alpha, \beta) = \min \left\{ \arcsin \beta, \arctan \left(\frac{\sin \pi\alpha}{\beta^{-\alpha} - \cos \pi\alpha} \right) \right\}.$$

□

3. PROOF OF THE GENERALIZED SECTOR THEOREM

In this section we prove Theorem 1. In the proof C_0, C_1, \dots are positive constants depending only on the parameters α, s, K, λ of the statement. Regarding hypothesis (b), let j_n be the smallest i in the interval $i_n \leq i < i_{n+1}$ such that $|a_i - a| = d_n$.

Start with any point $z_1 \in \mathbb{C}^+$, and set $z_{i+1} = A_i B_i(z_i)$ for $i = 1, 2, \dots, m$. Our goal is to show that $\theta(z_{m+1}) \geq \theta$, with θ as stated. Since we always have $\theta(z_2) \geq (1 - \alpha)\pi$, we may assume $m > 1$; otherwise there is nothing to prove. Let U_i , $i \geq 2$, be a square sitting in the upper half-plane, one of its sides being the interval $J_i^{\lambda/2} \subseteq J_i^\lambda$. We have $\text{diam } U_i = \sqrt{2}(1 + \frac{\lambda}{2})|J_i|$, while the distance between U_i and ∂I_i is $\geq \frac{\lambda}{4}|J_i|$. It follows from (1) and Koebe's distortion lemma that $N_{B_i}(U_i) \leq C_0$, where C_0 is a constant depending only on λ . Let V_i be the rectangle $U_i \setminus \{z : \text{Re } z > a\}$, and observe that $\theta(z) \leq \pi/2$ for all $z \in V_i$.

Lemma 5. *There exists $\theta_1 > 0$, depending only on α and λ , such that $A_i B_i(V_i)$ contains the triangle T_i in the upper half-plane with base $[A_i a_i, a] \subseteq \mathbb{R}$ and angles $\pi\alpha_i$ and θ_1 respectively at $A_i a_i$ and a .*

Proof. Since there exists $a < \xi_i < b$ such that $B_i'(\xi_i) = 1$ and the non-linearity of B_i over U_i is bounded by C_0 , we see from (2) that $C_1^{-1} \leq |B_i'(x)| \leq C_1$ for all $x \in J_i$, where $C_1 = e^{C_0}$. From hypothesis (c), we find $0 < C_2 < 1$ depending only on λ such that the disk D_i of radius $C_2|B_i^{-1}a_i - a|$ about $B_i^{-1}(a_i)$ is contained in V_i . Let $H_i \subseteq V_i$ be the Euclidean convex hull of $\{a\} \cup D_i$, and note that $H_i = \bigcup_{x \in [B_i^{-1}a_i, a]} D_x$, where D_x is the disk of center x and radius $r_x = C_2|x - a|$. By Koebe's one-quarter theorem, $B_i(D_x)$ contains the disk D'_x of center $B_i(x) \in [a_i, a]$ and radius

$$R_x = \frac{1}{4} r_x |B_i'(x)| \geq \frac{C_2}{4C_1} |x - a|.$$

Viewed from a , the disk D'_x comprises an angle $2\theta_x$, where

$$\sin \theta_x = \frac{R_x}{|B_i(x) - a|} \geq \frac{C_2}{4C_1^2}.$$

Therefore $B_i(H_i)$ contains the Euclidean convex hull H'_i of $\{a\} \cup D'_i$, where D'_i is the disk of center a_i and radius $(C_2/4C_1^2)|a_i - a|$. The result now follows if we apply Lemma 3 to $A = A_i$ and $W = H'_i \cap \mathbb{C}^+$, taking $\theta_1 = \inf_{0 < \alpha' \leq \alpha} \Theta(\alpha', C_2/4C_1^2) > 0$. \square

This lemma shows that, if $z \in \mathbb{C}^+ \setminus V_i$, then $\theta(A_i B_i(z)) \geq \theta_1$, for all $i \geq 2$. In what follows we assume, as we may, that $\theta_1 \leq \pi/4$. Let j be the smallest index ≥ 2 such that $z_i \in V_i$ for all $i \geq j$ (if no such j exists, then $\theta(z_{m+1}) \geq \theta_1$ and we are done). Take the smallest $j_n \geq j$ and note that $t = j_n - j \leq 2s$, from hypothesis (b). Our aim is to control the *loss of angle* when we apply to z_j the next t factors $A_j B_j, A_{j+1} B_{j+1}, \dots, A_{j_n-1} B_{j_n-1}$.

As we know, z_j belongs to $V_j^* = V_j \setminus \{z : 0 \leq \theta(z) < \theta_1\}$. Let Q_j be the square of Figure 2 containing V_j^* . We have $\text{diam } Q_j \leq \sqrt{2} \text{diam } V_j \leq C_3|a_j - a|$. Moreover, by hypothesis (c), the *base* of $\phi_j = A_j B_j$ contains intervals of length $\geq C_4|a_j - a|$ on both sides of (a, b) , whence the distances from its endpoints to Q_j are $\geq C_5|a_j - a| \sin \theta_1$ (this uses the fact that $\sin \theta_1 \leq \cos \theta_1$, coming from $\theta_1 \leq \pi/4$). Hence, $N_{\phi_j}(Q_j) \leq C_6(\sin \theta_1)^{-1}$, again by (1) and Koebe's distortion

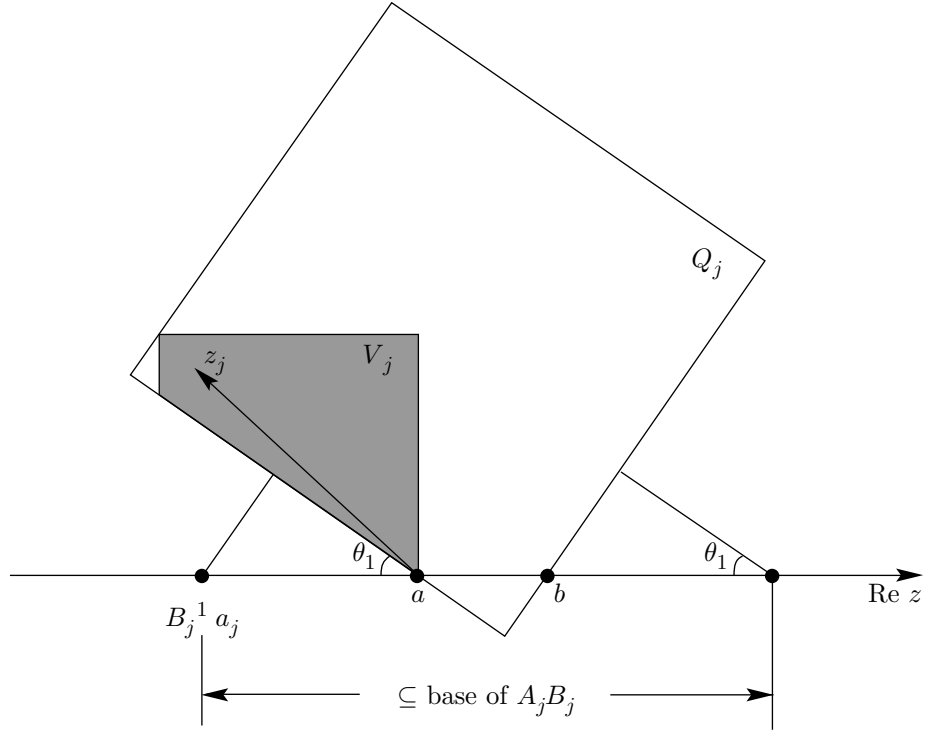


FIGURE 2

lemma. Therefore, by Lemma 2, we have

$$(3) \quad \sin \theta(z_{j+1}) \geq \exp\{-C_6(\sin \theta_1)^{-1}\} \sin \theta(z_j) .$$

Defining $0 < \theta_k < \theta_1$ recursively by

$$\sin \theta_{k+1} = \exp\{-C_6(\sin \theta_k)^{-1}\} \sin \theta_k ,$$

for $k = 2, \dots, t$, and repeating the argument leading to (3) another $t - 1$ times, we get $\theta(z_{j_n}) \geq \theta_t$, which is a constant still depending only on the parameters in the statement.

Now we appeal to Poincaré-neighborhood trapping in order to control the loss of angle due to the remaining factors. Let \mathcal{P} be the smallest of the Poincaré neighborhoods $\mathcal{P}(r)$ of (a, b) containing $V_{j_n} \setminus \{z : 0 \leq \theta(z) < \theta_t\}$, and let $\mathcal{P}^+ = \mathcal{P} \cap \mathbb{C}^+$. From Lemma 3 we know that $2R = \text{diam } \mathcal{P}^+ = O(d_n)$. Note that since $\theta_t \leq \pi/4$, we have $R \geq |a - b|/\sqrt{2}$, which is used in the proof of the following lemma.

Lemma 6. *For each $i \geq j_n$, we have $\text{dist}(a_i, \mathcal{P}^+) \geq C_7|a_i - a|$.*

Proof. Let h be the distance from the center of \mathcal{P}^+ to the real axis, let Δ_i be the distance from a_i to the midpoint of (a, b) , and let $\delta_i = \text{dist}(a_i, \mathcal{P}^+)$. Then

elementary geometry gives us

$$(4) \quad \frac{\delta_i}{|a_i - a|} \geq \frac{\delta_i}{\Delta_i} = \frac{1 - \left(\frac{|a-b|}{2\Delta_i}\right)^2}{\frac{R}{\Delta_i} + \sqrt{1 + \left(\frac{h}{\Delta_i}\right)^2}}.$$

By hypothesis (b) we have $|a_i - a| \geq K^{-1}d_n \geq C_8R$ for all $i \geq j_n$. Therefore $h < R \leq C_8^{-1}\Delta_i$, and this bounds the denominator of (4) from above. At the same time,

$$\Delta_i = |a_i - a| + \frac{1}{2}|a - b| \geq (1 + C_8\sqrt{2})\frac{|a - b|}{2},$$

which in turn bounds the numerator of (4) from below. \square

For each $i \geq j_n$, let $(x_i^+, x_i^-) \subseteq \mathbb{R}$ be the base of $\phi_i = A_iB_i$. By hypothesis (c), we have $|x_i^- - a| \geq C_9|a_i - a|$ and $|x_i^+ - b| \geq C_9|a_i - a|$. Hence the same method used in the proof of Lemma 6 gives also

$$(5) \quad \text{dist}(x_i^\pm, \mathcal{P}^+) \geq C_{10}|a_i - a|.$$

Therefore, once again by Koebe's distortion lemma, we see that

$$(6) \quad N_{\phi_i}(\mathcal{P}^+) \leq C_{11}d_n/|a_i - a|.$$

Since $\phi_i(\mathcal{P}^+) \subseteq \mathcal{P}^+$ for all $i \geq j_n$, applying Lemma 2 recursively together with (6) yields

$$(7) \quad \begin{aligned} \sin \theta(z_{m+1}) &\geq \left(\prod_{i=j_n}^m \exp \{ -N_{\phi_i}(\mathcal{P}^+) \} \right) \sin \theta(z_{j_n}) \\ &\geq \exp \left\{ -C_{11}d_n \sum_{i=j_n}^m |a_i - a|^{-1} \right\} \sin \theta(z_{j_n}). \end{aligned}$$

But here, using the full strength of hypothesis (b), we have

$$\sum_{i=j_n}^m |a_i - a|^{-1} \leq s \sum_{k \geq n} d_k^{-1} \leq Ksd_n^{-1}.$$

Taking this back to (7), we get at last $\sin \theta(z_{m+1}) \geq \exp\{-C_{11}Ks\} \sin \theta_t$. This completes the proof of Theorem 1.

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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, RUA DO MATÃO 1010, BUTANTÃ CEP05508-900 SÃO PAULO SP, BRASIL

E-mail address: edson@ime.usp.br