

## ON THE ASYMPTOTICITY ASPECT OF HYERS-ULAM STABILITY OF MAPPINGS

D. H. HYERS, G. ISAC, AND TH. M. RASSIAS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The object of the present paper is to prove an asymptotic analogue of Th. M. Rassias' theorem obtained in 1978 for the Hyers-Ulam stability of mappings.

### 1. INTRODUCTION

In [15] Rassias generalized the result of Hyers [9] by allowing growth of the form  $\varepsilon \cdot (\|x\|^p + \|y\|^p)$  for the norm of the Cauchy difference  $f(x+y) - f(x) - f(y)$ , where  $0 \leq p < 1$ , and still obtained the formula

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

for the additive mapping approximating  $f$ . Other developments of this idea are described in [10] (see also [1], [5], [7], [8], [12], [13], [16]). In the present article we obtain an asymptotic analogue of this result of Th. M. Rassias.

Several authors have used asymptotic conditions in stating approximations to Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

P.D.T.A. Elliott [6] showed that if the real function  $f$  belongs to the class  $L^p(0, z)$  for every  $z \geq 0$ , where  $p \geq 1$ , and satisfies the asymptotic condition

$$\lim_{z \rightarrow \infty} \frac{\int_0^z \int_0^z |f(x+y) - f(x) - f(y)|^p dx dy}{z} = 0,$$

then there is a constant  $c$  such that  $f(x) = cx$  almost everywhere on  $\mathbb{R}^+$ . One of the theorems of J. R. Alexander, C. E. Blair and L. A. Rubel [1] states that if  $f \in L^1(0, b)$  for all  $b > 0$ , and if for almost all  $x > 0$

$$\lim_{u \rightarrow \infty} \frac{\int_0^u [f(x+y) - f(x) - f(y)] dy}{u} = 0,$$

then for some real number  $c$ ,  $f(x) = cx$  for almost all  $x \geq 0$ .

---

Received by the editors December 11, 1995 and, in revised form, May 21, 1996 and July 29, 1996.

1991 *Mathematics Subject Classification*. Primary 39B72, 47H15.

*Key words and phrases*. Hyers-Ulam stability, asymptotic conditions, asymptotically derivable, additive outside a ball.

F. Skof [17] proved that given real normed spaces  $X$  and  $E$  and a mapping  $f : X \rightarrow E$  satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \rightarrow 0 \quad \text{as } \|x\| + \|y\| \rightarrow \infty,$$

then  $f(x+y) = f(x) + f(y)$  for all  $x$  and  $y$  in  $X$ . In a later article [18] the same author showed that a real-valued function  $f$  defined on a real normed space  $X$  is additive proving that  $f(0) = 0$  and  $|f(x+y) - f(x) - f(y)| \rightarrow 0$  when  $\|x\| + \|y\| \rightarrow 0$ . In [12] is shown an interesting relation between the Hyers-Ulam stability and the asymptotic derivability. This relation is applied to the study of some important nonlinear problems (cf. [13]).

In the present paper we consider the asymptotic aspect of Hyers-Ulam stability close to the asymptotic derivability. The asymptotic derivability is very important in nonlinear analysis (cf. [2], [3], [4], [11], [14]).

## 2. MAIN RESULT

**Theorem 1.** *Given a real normed vector space  $E_1$  and a real Banach space  $E_2$ , let numbers  $M > 0, \varepsilon > 0$  and  $p$  with  $0 < p < 1$  be chosen. Let the mapping  $f : E_1 \rightarrow E_2$  satisfy the inequality*

$$(1) \quad \|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all  $x, y$  in  $E_1$  such that

$$(2) \quad \|x\|^p + \|y\|^p > M^p.$$

Then there exists an additive mapping  $\varphi : E_1 \rightarrow E_2$  such that

$$(3) \quad \|\varphi(x) - f(x)\| < \beta(p)\varepsilon\|x\|^p$$

for all  $x \in E_1$  with  $\|x\| > \frac{M}{2^{1/p}}$ , where  $\beta(p) = \frac{2}{2-2^p}$  and  $\varphi(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ .

*Proof.* When  $\|x\| > \frac{M}{2^{1/p}}$ , that is, when  $2\|x\|^p > M^p$ , we may put  $y = x$  in (1) to obtain

$$(4) \quad \|2^{-1}f(2x) - f(x)\| \leq \varepsilon\|x\|^p.$$

Of course we can replace  $x$  by  $2x$  in (4) since  $\|2x\|$  is also greater than  $\frac{M}{2^{1/p}}$ . Thus, we can use the argument given in [15] to arrive at the inequality

$$(5) \quad \|2^{-n}f(2^n x) - f(x)\| \leq \beta(p)\varepsilon\|x\|^p \quad \text{when } \|x\| > \frac{M}{2^{1/p}} \text{ for } n \in N$$

and thus to show that the limit

$$(6) \quad g(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists when  $\|x\| > \frac{M}{2^{1/p}}$ . Therefore

$$(7) \quad \|g(x) - f(x)\| \leq \beta(p)\varepsilon\|x\|^p.$$

Clearly, when  $\|x\| > \frac{M}{2^{1/p}}$ ,  $g(2x) = \lim_{n \rightarrow \infty} \frac{f(2^{n+1}x)}{2^n} = 2 \lim_{n \rightarrow \infty} \frac{f(2^{n+1}x)}{2^{n+1}}$ , so that

$$(8) \quad g(2x) = 2g(x) \quad \text{for } \|x\| > \frac{M}{2^{1/p}}.$$

Now suppose that  $\|x\|, \|y\|$  and  $\|x+y\|$  are all greater than  $\frac{M}{2^{1/p}}$ . Then by (1) we find that for all  $n \in N$ ,

$$(9) \quad \|2^{-n}f(2^n(x+y)) - 2^{-n}f(2^n x) - 2^{-n}f(2^n y)\| \leq \varepsilon 2^{-n(1-p)}(\|x\|^p + \|y\|^p).$$

Under the conditions stated it follows by (6) that

$$(9a) \quad g(x+y) = g(x) + g(y).$$

Using an extension method of F. Skof [18] we will define a mapping  $\varphi : E_1 \rightarrow E_2$  to be an extension of the mapping  $g$  to the whole space  $E_1$ . Given any  $x \in E_1$  with  $0 < \|x\| < \frac{M}{2^{1/p}}$ , let  $k = k(x)$  denote the largest integer such that

$$(10) \quad \frac{M}{2^{1/p}} < 2^k \|x\| \leq M.$$

Define the mapping  $\varphi$  as follows:

$$\begin{cases} \varphi(0) = 0, \\ \varphi(x) = 2^{-k} g(2^k x) & \text{for } 0 < \|x\| < \frac{M}{2^{1/p}}, \text{ where } k = k(x), \\ \varphi(x) = g(x) & \text{for } \|x\| > \frac{M}{2^{1/p}}. \end{cases}$$

**Lemma.** For all  $x$  in  $E_1$

$$(11) \quad \varphi(x) = \lim_{s \rightarrow \infty} 2^{-s} f(2^s x)$$

and

$$(12) \quad \varphi(-x) = -\varphi(x).$$

*Proof.* Take any  $x$  in  $E_1$  with  $0 < \|x\| < \frac{M}{2^{1/p}}$ , and let  $k = k(x)$ , so that  $k$  is the largest integer satisfying (10). Thus  $k-1$  is the largest integer satisfying

$$\frac{M}{2^{1/p}} < \|2^{k-1}(2x)\| \leq M,$$

and we have

$$\varphi(2x) = 2^{-(k-1)} g(2^{k-1}(2x)) = 2^{-k} \cdot 2g(2^k x) = 2\varphi(x) \quad \text{for } 0 < \|x\| < \frac{M}{2^{1/p}}.$$

From the definition of  $\varphi$  and property (8) of  $g$  it follows that  $\varphi(2x) = 2\varphi(x)$  for all  $x$  in  $E_1$ . Given  $x$  in  $E_1$  with  $x \neq 0$ , choose a positive integer  $m$  so large that  $\|2^m x\| > \frac{M}{2^{1/p}}$ .

By the definition of  $\varphi$  we have

$$\varphi(x) = 2^{-m} \varphi(2^m x) = 2^{-m} g(2^m x),$$

and by (6) this implies that

$$\varphi(x) = \lim_{n \rightarrow \infty} 2^{-(m+n)} f(2^{m+n} x),$$

which demonstrates (11) for  $x \neq 0$ .

Since  $\varphi(0) = 0$ , the same is true for  $x = 0$ . Equation (12) is obvious for  $x = 0$ . Take any  $x$  in  $E_1$  with  $x \neq 0$  and choose  $n \in \mathbb{N}$  large enough so that  $\|2^n x\| > \frac{M}{2^{1/p}}$ .

Then by (1) with  $y = -x$  we obtain

$$\|2^{-n} f(2^n x) + 2^{-n} f(-2^n x)\| \leq 2\epsilon 2^{-n(1-p)} \|x\|^p + 2^{-n} \|f(0)\|.$$

When  $n \rightarrow \infty$  it follows from (11) that (12) holds. The lemma is proved.

In proving the additivity of  $\varphi$  we note that the equation

$$(13) \quad \varphi(x+y) = \varphi(x) + \varphi(y)$$

holds when either  $x$  or  $y$  is zero.

Assume then that  $x \neq 0$  and  $y \neq 0$ . If  $x+y=0$ , i.e.  $y=-x$ , then (12) shows that (13) holds. The only remaining case is when  $x, y$  and  $x+y$  are all different from

zero. In this case we may choose an  $n$  in  $N$  such that  $\|2^n x\|$ ,  $\|2^n y\|$  and  $\|2^n(x+y)\|$  are all greater than  $\frac{M}{2^{1/p}}$ . Then by (1) we have

$$\|f(2^n(x+y)) - f(2^n x) - f(2^n y)\| \leq \varepsilon 2^{np}(\|x\|^p + \|y\|^p).$$

If we divide both sides of this inequality by  $2^n$  and then let  $n \rightarrow \infty$ , we find by (11) that (13) is true, thus  $\varphi$  is additive.

By definition  $\varphi(x) = g(x)$  when  $\|x\| > \frac{M}{2^{1/p}}$ , thus (3) follows from (7) and the proof of Theorem 1 is complete. Q.E.D.

For convenience in applications we give the following modified version of Theorem 1.

**Theorem 2.** *Given a real normed vector space  $E_1$  and a real Banach space  $E_2$ , let numbers  $m > 0, \varepsilon > 0$  and  $p$  with  $0 \leq p < 1$  be chosen. Suppose that the mapping  $f : E_1 \rightarrow E_2$  satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all  $x$  and  $y$  in  $E_1$  such that  $\|x\| > m, \|y\| > m$  and  $\|x+y\| > m$ . Then there exists an additive mapping  $\varphi : E_1 \rightarrow E_2$  which satisfies

$$\|\varphi(x) - f(x)\| \leq 2\varepsilon(2 - 2^p)^{-1}\|x\|^p$$

for all  $x$  in  $E_1$  such that  $\|x\| > m$ . Moreover,  $\varphi$  is given by the formula

$$\varphi(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

for all  $x$  in  $E_1$ .

*Proof.* Assume that  $\|x\| > m$ . Then as in the proof of Theorem 1 we obtain (4)–(8) inclusive, but now all these formulas are satisfied for  $\|x\| > m$ . In particular,

$$g(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x) \quad \text{when } \|x\| > m.$$

Also, if  $\|x\| > m, \|y\| > m$  and  $\|x+y\| > m$ , then by hypothesis we see that (9) and (9a) also hold. To apply Skof's extension procedure in the present case, let  $x$  in  $E_1$  be given with  $0 < \|x\| \leq m$  and define  $k = k(x)$  to be the unique positive integer such that

$$(14) \quad m < 2^k \|x\| \leq 2m.$$

Now define the mapping  $\varphi : E_1 \rightarrow E_2$  as follows:

$$\begin{cases} \varphi(0) = 0, \\ \varphi(x) = 2^{-k} g(2^k x) & \text{for } 0 < \|x\| \leq m, \\ \varphi(x) = g(x) & \text{for } \|x\| > m. \end{cases}$$

The proof of the Lemma used in the proof of Theorem 1, follows as before with the obvious changes.

Indeed, we start with  $x$  in  $E_1$  satisfying  $0 < \|x\| \leq m$  and let  $k = k(x)$  as defined by (14), etc. Thus the Lemma holds under the conditions of Theorem 2. The proof of the additivity of  $\varphi$  also follows as before. Therefore the proof of Theorem 2 is complete. Q.E.D.

3.  $p$ -ASYMPTOTICAL ADDITIVITY

We apply the main theorem, precisely Theorem 2, to the study of  $p$ -asymptotical derivatives.

Let  $E_1$  and  $E_2$  be Banach spaces. Let  $T$  be a mapping from  $E_1$  into  $E_2$  satisfying eventually a special property such as, for example, additivity, linearity, etc. Let  $0 < p < 1$  be arbitrary.

**Definition 1.** A mapping  $f : E_1 \rightarrow E_2$  is  $p$ -asymptotically close to  $T$  if and only if  $\lim_{\|x\| \rightarrow \infty} \frac{\|f(x) - T(x)\|}{\|x\|^p} = 0$ .

*Remark 1.* If in Definition 1,  $T \in L(E_1, E_2)$ , then we say that  $T$  is a  $p$ -asymptotical derivative of  $f$  and if such a  $T$  exists, then  $f$  is  $p$ -asymptotically derivable.

*Remark 2.* Since for  $x$  such that  $\|x\| \geq 1$  we have  $\|x\|^p \leq \|x\|$ , one obtains that every  $p$ -asymptotical derivative of  $f$  is an asymptotical derivative. Indeed, if  $T \in L(E_1, E_2)$  is a  $p$ -asymptotical derivative of  $f$ , then

$$0 \leq \lim_{\substack{\|x\| \rightarrow \infty \\ \|x\| > 1}} \frac{\|f(x) - T(x)\|}{\|x\|} \leq \lim_{\substack{\|x\| \rightarrow \infty \\ \|x\| > 1}} \frac{\|f(x) - T(x)\|}{\|x\|^p} = 0.$$

**Definition 2.** A mapping  $f : E_1 \rightarrow E_2$  is  $p$ -asymptotically additive if and only if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all  $x, y \in E$  such that  $\|x\|^p, \|y\|^p, \|x+y\|^p > \delta$ .

**Definition 3.** A mapping  $T : E_1 \rightarrow E_2$  is additive outside a ball if there exists  $r > 0$ , such that  $T(x+y) = T(x) + T(y)$  for all  $x, y \in E_1$  with  $\|x\|, \|y\| \geq r$  and  $\|x+y\| \geq r$ .

**Example.** Let  $T : E_1 \rightarrow E_2$  be defined by

$$T(x) = \begin{cases} L(x) & \text{if } \|x\| \geq r, \\ \varphi(x) & \text{if } \|x\| < r \end{cases}$$

where  $L : E_1 \rightarrow E_2$  is a linear mapping and

$$\varphi : B(0, r) \rightarrow E_2$$

is a nonlinear mapping where  $B(0, r) = \{x \in E_1 \mid \|x\| < r\}$ . It follows that if  $x, y \in E_1$  with  $\|x\| \geq r, \|y\| \geq r$ , and  $\|x+y\| \geq r$ , then  $T(x+y) = T(x) + T(y)$ .

We have the following result.

**Theorem 3.** If  $f : E_1 \rightarrow E_2$  is  $p$ -asymptotically close to an additive mapping outside a ball  $T : E_1 \rightarrow E_2$ , then  $f$  is  $p$ -asymptotically additive.

**Theorem 4.** If  $f : E_1 \rightarrow E_2$  is  $p$ -asymptotically close to an additive outside a ball mapping  $T : E_1 \rightarrow E_2$ , then  $f$  is  $p_*$ -asymptotically close to an additive mapping, where  $0 < p < p_* < 1$ .

**Corollary.** If  $f : E_1 \rightarrow E_2$  is  $p$ -asymptotically close to an additive outside a ball mapping  $T : E_1 \rightarrow E_2$ , then  $f$  has an additive asymptotical derivative.

## ACKNOWLEDGMENT

We wish to express our thanks to the referees for their helpful remarks.

## REFERENCES

1. R. Alexander, C. E. Blair and L. A. Rubel, *Approximate version of Cauchy's functional equation*, Illinois J. Math. **39** (1995), 278–287. MR **96b**:39018
2. H. Amann, *Fixed points of asymptotically linear maps in ordered Banach spaces*, J. Funct. Anal. **14** (1973), 162–171. MR **50**:3019
3. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), 620–709. MR **54**:3519
4. N. P. Câc and J. A. Gatica, *Fixed point theorems for mappings in ordered Banach spaces*, J. Math. Anal. Appl. **71** (1979), 547–557. MR **80j**:47071
5. J. Chmieliński, *On the stability of the generalized orthogonality equation*, In: Stability of Mappings of Hyers-Ulam Type (eds. Th. M. Rassias & J. Tabor), Hadronic Press, Palm Harbour, Florida, 1994, pp. 31–57. MR **95j**:39046
6. P. D. T. A. Elliott, *Cauchy's functional equation in the mean*, Advances in Math. **51** (1984), 253–257. MR **86d**:39009
7. R. Ger, *On functional inequalities stemming from stability questions*, In: General Inequalities 6, (ed. W. Walter), Internat. Ser. Numer. Math., Vol. 103, Birkhäuser Verlag, Basel, 1992, pp. 227–240. MR **94b**:39042
8. R. Ger and J. Sikorska, *Stability of the orthogonal additivity*, Bull. Polish Acad. Sci. Math. **43** (1995), 143–151. CMP 97:03
9. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224. MR **2**:315a
10. D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequat. Math. **44** (1992), 125–153. MR **93i**:39007
11. G. Isac, *Opérateurs asymptotiquement linéaires sur des espaces localement convexes*, Colloq. Math. **46** (1982), 67–72. MR **84h**:58014
12. G. Isac and Th. M. Rassias, *On the Hyers-Ulam stability of  $\psi$ -additive mappings*, J. Approx. Theory **72** (1993), 131–137. MR **94b**:39043
13. G. Isac and Th. M. Rassias, *Stability of  $\psi$ -additive mappings: Applications to nonlinear analysis*, Intern. J. Math. & Math. Sciences **19** (1996), 219–228. MR **96m**:47114
14. M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, (1964). MR **31**:6107
15. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300. MR **80d**:47094
16. Th. M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338. MR **94d**:39011
17. F. Skof, *Sull' approssimazione delle applicazioni localmente  $\delta$ -additive*, Atti. Accad. Sc. Torino **117** (1983), 377–389. MR **89a**:39015
18. F. Skof, *On the stability of functional equations on a restricted domain and a related topic*, In: Stability of Mappings of Hyers-Ulam type (eds. Th. M. Rassias and J. Tabor), Hadronic Press, Palm Harbor, Florida, 1994, pp. 141–151. MR **96a**:39035

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089-1113

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ROYAL MILITARY COLLEGE OF CANADA, KINGSTON, ONTARIO, CANADA K7K 5L0

DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY OF ATHENS, ZOGRAFOU CAMPUS, 15780 ATHENS, GREECE