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# ON THE INFINITE PRODUCT OF OPERATORS IN HILBERT SPACE

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ABSTRACT. We give a necessary and sufficient condition for a certain set of infinite products of linear operators to be zero. We shall investigate also the case when this set of infinite products converges to a non-zero operator.

The main device in these results is a weighted version of the König Lemma for infinite trees in graph theory.

The subject of this paper evolves around the following problem:

*Let* $\{A_k;\, k = 1, 2, \dots, N\}$ *be a finite set of bounded linear operators in a Hilbert space* $\mathcal{H}$*. Suppose that the sequence* $\{A_k^n;\, n = 1, 2, \dots\}$ *is convergent for every* $k$*. What are the conditions for the convergence of the infinite product*

$$\text{(P)} \qquad \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}, \qquad \sigma i \in \{1, 2, \dots, N\}?$$

In a natural way the infinite product (P) corresponds to the recursive sequence

$$x_0 \in \mathcal{H}, \qquad x_{n+1} = A_{\sigma n} x_n, \quad n = 0, 1, \dots .$$

This paper was motivated by Daubechies and Lagarias [1] where a similar problem was solved for the infinite product of $n \times n$ matrices. (See also [3].) However, our results are different from the results in [1] even in the case when $\mathcal{H} = \mathbb{R}^n$. In [1], conditions for convergence are given for *every* infinite product (P) from a finite set $\{A_i;\, i = 1, 2, \dots, N\}$ of matrices. Our results have a local nature; we find convergence conditions for (P) *only for certain subsets* $\mathcal{J}$ *of labels* $\sigma_1\sigma_2 \cdots \sigma_n \cdots$.

Moreover, lifting a part of the proof of [1], Theorem 3.1, to a general theorem called the Weighted König Lemma, the essence of the reasonings have been emphasized by "unfolding" the essential idea of this theorem.

Our problem and results are closely related to ergodic theory and the key object $\hat{\phi}$ in our investigations (see Proposition 1, resp. Section 4) is essentially the same as (1.1) in [3], Theorem 1.1.

## 1. PRELIMINARIES

Throughout the paper $\mathcal{H}$ will mean a Hilbert space and $B(\mathcal{H})$ the usual normed algebra of bounded linear operators of $\mathcal{H}$.

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Let $\mathcal{H} = F \oplus V$ be an orthogonal decomposition of $\mathcal{H}$ and $P_F, P_V$ be the corresponding projections. For $A \in B(\mathcal{H})$ we define the decomposition of $A$ corresponding to $\mathcal{H} = F \oplus V$ as follows.

Considering the operator matrix

$$\text{(OH)} \qquad \begin{pmatrix} P_F A P_F & P_F A P_V \\ P_V A P_F & P_V A P_V \end{pmatrix}$$

it is easy to verify that

$$Ax = \begin{pmatrix} P_F A & P_F A \\ P_V A & P_V A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{where } u = P_F x, v = P_V x.$$

*The operator matrix* (OH) *will be called the decomposition of* $A$ *corresponding to* $\mathcal{H} = F \oplus V$.

If $Au = u$ for every $u \in F$, i.e. if every $u \in F$ is a fixed point of $A$, then it is easy to verify that $AP_F = P_F$ and $P_V A P_F = P_V P_F = 0$. Hence the decomposition (OH) in this case is

$$\text{(M)} \qquad A = \begin{pmatrix} P_F & \mathfrak{C} \\ 0 & \mathfrak{U} \end{pmatrix}$$

where $\mathfrak{C} = P_F A P_V$ and $\mathfrak{U} = P_V A P_V$. *Throughout the paper we shall use these gothic notations for these restrictions of an* $A \in B(\mathcal{H})$.

Let $\mathcal{J}$ be a set of infinite strings

$$\sigma := \sigma_1 \sigma_2 \cdots \sigma_n \cdots$$

of $N$ symbols, e.g. $\{1, 2, \dots, N\}$. $\mathcal{J}$ is a metric space with the metric

$$d_c(\sigma, \omega) := \sum_{k=1}^{\infty} \frac{|\sigma_k - \omega_k|}{(N+1)^k}, \qquad \sigma, \omega \in \mathcal{J}.$$

We have a natural mapping from $\mathcal{J}$ into a set $\mathcal{S}$ of infinite product of operators. Let

$$\{A_k;\ k = 1, 2, \dots, N\}, \qquad A_k \in B(\mathcal{H}),$$

be given. Then the image of $\sigma := \sigma_1 \sigma_2 \cdots \sigma_n \cdots$ is the infinite product

$$M(\sigma) := \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} \quad \text{with } A_{\sigma i} \in \{A_k;\ k = 1, 2, \dots, N\}.$$

If each $M(\sigma) \in \mathcal{S}$ is convergent in the operator norm, then $\mathcal{S}$ is a metric space with the operator norm and then we can speak about the continuity of the function $M = M(\sigma)$.

## 2. On the convergence of $\{A^n;\ n = 1, 2, \dots\}$

**Lemma 1.** *Let* $A$ *be a bounded linear operator in a Hilbert space* $\mathcal{H}$ *and* $\rho(A) = \limsup_n \|A^n\|^{1/n}$ *(i.e. the spectral radius of* $A$*). Then*

$$A^n \Rightarrow 0 \quad \textit{if and only if} \quad \rho(A) < 1.$$

*Proof.* If $\rho(A) < 1$, then there exists an $r$ such that $\|A^r\|^{1/r} \le s < 1$ and hence for $k > r$

$$\|A^k\| < M^r s^w$$

where $M = \max\{1, \|A\|\}$ and $w$ is an integer in $(k - r, k]$.

It follows that $A^n \Rightarrow 0$ in an exponential rate.

If $\rho(A) \geq 1$, then $A^n$ does not tend to zero. Indeed, $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ (see e.g. [2], VII.3.4) and

$$\|A^{2n}\|^{1/2n} \leq \|A^n\|^{1/2n}\|A^n\|^{1/2n} = \|A^n\|^{1/n}, \qquad n = 1, 2, \dots .$$

Hence the subsequence $\{\|A^n\|^{1/n}; n = 2^m,\ m = 1, 2, \dots\}$ is decreasing. It follows that

$$1 \leq \rho(A) \leq \|A^n\|^{1/n} \quad \text{for } n = 2^m.$$

**Lemma 2.** *Let $A$ be a bounded linear operator of $\mathcal{H}$ and assume that there exists an orthogonal decomposition*

$$\mathcal{H} = F \oplus V \tag{H}$$

*such that the* (OH) *decomposition corresponding to* (H) *is in the form* (M) *with $\rho(\mathfrak{U}) < 1$.*

*Then $\{A^n;\ n = 1, 2, \dots\}$ is convergent. Moreover, if $A^n \Rightarrow C \neq 0$, then the* (OH) *decomposition of $C$ is*

$$\begin{pmatrix} P_F & \mathfrak{C}(I - \mathfrak{U})^{-1} \\ 0 & 0 \end{pmatrix} \tag{C}$$

*where $I$ is the identity.*

*Proof.* It follows from Lemma 1 that $\mathfrak{U}^n \Rightarrow 0$. Moreover,

$$A^n x = \begin{pmatrix} P_F & \mathfrak{C}(I + \mathfrak{U} + \dots + \mathfrak{U}^{n-1}) \\ 0 & \mathfrak{U}^n \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \mathfrak{C}(I + \mathfrak{U} + \dots + \mathfrak{U}^{n-1})v \\ \mathfrak{U}^n v \end{pmatrix}$$

by induction from (M). (C) follows from straightforward calculation since $\rho(\mathfrak{U}) < 1$.

**Lemma 3.** *If $A^n \Rightarrow C \neq 0$, then the limit of $\{A^n x;\ n = 1, 2, \dots\}$ is a fixed point of $A$ for every $x \in \mathcal{H}$.*

*Let $F(A)$ be the closed linear subspace of the fixed points of $A$ and consider the orthogonal decomposition $\mathcal{H} = F(A) \oplus V$. Then the corresponding* (OH) *decomposition of $A$ is*

$$\begin{pmatrix} P_F & \mathfrak{C} \\ 0 & \mathfrak{U} \end{pmatrix}$$

*where $\mathfrak{U}^n \Rightarrow 0$ and hence $\rho(\mathfrak{U}) < 1$.*

*Proof.* $\{A^n x;\ n = 1, 2, \dots\}$ and $\{A^{n-1}x;\ n = 1, 2, \dots\}$ have the same limit $z$, i.e.

$$z = \lim A^n x = \lim A(A^{n-1}x) = Az.$$

$P_F A P_F = P_F$ since every $u \in F(A)$ is a fixed point, moreover, in this case $P_V A P_F = P_V P_F = 0$.

For $\mathfrak{U}^n \Rightarrow 0$ we have to show that 0 is the only fixed point of $\mathfrak{U}$. If

$$\mathfrak{U}z = P_V A P_V z = z,$$

then $z \in V$ and hence

$$Az = AP_V z = P_V A P_V z + P_F A P_V z = z + u^*, \qquad u^* \in F(A).$$

Then

$$A^n z = z + n \cdot u^*$$

follows by induction. Since $\{A^n;\ n = 1, 2, \dots\}$ is convergent, it follows that $u^* = 0$. Moreover, $z = 0$ since $z \in V$ and $V$ is the orthogonal complement of $F(A)$.

## 3. The Weighted König Lemma

Let $\mathcal{J}$ be a subset of infinite strings $\sigma := \sigma_1\sigma_2\cdots\sigma_n\cdots$ of $N$ symbols and $S\sigma = \sigma_2\sigma_3\cdots\sigma_{n+1}\cdots$. $\mathcal{J}$ is called *shift-invariant* if

$$\sigma \in \mathcal{J} \quad \text{implies} \quad S\sigma \in \mathcal{J}.$$

The *finite* string $\sigma_1\sigma_2\cdots\sigma_k$ is called a *prefix* of $\sigma' \in \mathcal{J}$ if $\sigma'_i = \sigma_i$ for $i \leq k$.

Let $\Sigma$ be the set of prefixes of $\mathcal{J}$ and let $\phi : \Sigma \Rightarrow \mathbb{R}^+$ be a weight function with the property

$$\phi(\sigma_1\sigma_2\cdots\sigma_k\sigma_{k+1}\cdots\sigma_m) \leq \phi(\sigma_1\sigma_2\cdots\sigma_k)\cdot\phi(\sigma_{k+1}\sigma_{k+2}\cdots\sigma_m). \tag{1}$$

If $m = 1$ and $\sigma_1 = i$, then we shall use the shorter notation $\phi(\sigma_1) := \phi(i)$.

Our motivation in this concept is the following

**Lemma 4.** *Let us consider the following properties*:

I. $\mathcal{J}$ *is shift-invariant*;
II. *If* $\sigma_1\sigma_2\cdots\sigma_k \in \Sigma$, *then every part of the prefix* $\sigma_1\sigma_2\cdots\sigma_k$ *also belongs to* $\Sigma$, *i.e.* $\sigma_m\sigma_{[m+1]}\cdots\sigma_1 \in \Sigma$ *for every* $1 \leq m \leq 1 \leq k$;
III. (1) *is defined for every prefix in* $\Sigma$.

*Then* I$\Rightarrow$II$\Rightarrow$III.

*Proof.* Obvious.

The next *Proposition* is the central part of this Section, however, seemingly it has little to do with graph theory at the first sight.

**Proposition 1.** *Let*

$$\hat{\phi}_k = \max\{\phi(\sigma_1\sigma_2\cdots\sigma_k); \sigma \in \mathcal{J}\} \quad \textit{and} \quad \hat{\phi} = \limsup\{\hat{\phi}_k^{1/k};\ k = 1, 2, \ldots\}.$$

*Then*

$$\mathcal{T} = \{\sigma_1\sigma_2\cdots\sigma_n : \phi(\sigma_1\sigma_2\cdots\sigma_k) \geq \hat{\phi}^k \textit{ for } k \leq n\}$$

*is an infinite subset of* $\Sigma$.

*Proof.* $\mathcal{T}$ is not empty since

$$\max\{\phi(i);\ i = 1, 2, \ldots, N\} \geq \hat{\phi}$$

follows from (1).

We shall suppose that $\mathcal{T}$ is a finite set and will arrive at a contradiction.

*Step* 1. Let us define the *boundary* $\mathcal{C}$ *of* $\mathcal{T}$ as follows:
for $k > 1$: $\sigma_1\sigma_2\cdots\sigma_k \in \mathcal{C}$ if $\sigma_1\sigma_2\cdots\sigma_k \notin \mathcal{T}$ and $\sigma_1\sigma_2\cdots\sigma_{k-1} \in \mathcal{T}$,
for $k = 1$: $\sigma_1 \in \mathcal{C}$ if $\sigma_1 \notin \mathcal{T}$.

If $\mathcal{T}$ is finite, then $\mathcal{C}$ is also finite. Moreover

$$\phi(\sigma_1\sigma_2\cdots\sigma_k)^{1/k} < \hat{\phi} \quad \text{for } \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{C}$$

and hence there exists an $\alpha > 0$ such that

$$\max\{\phi(\sigma_1\sigma_2\cdots\sigma_k)^{1/k};\ \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{C}\} = \hat{\phi} - \alpha \tag{2}$$

since $\mathcal{C}$ is finite.

*Step* 2. If $r$ is the length of the longest prefix in $\mathcal{C}$, then every $\sigma \in \mathcal{J}$ is the concatenation of finite strings belonging to $\mathcal{C}$ of length of at most $r$. Hence for every prefix $\sigma_1\sigma_2\cdots\sigma_k$ of $\sigma \in \mathcal{J}$ with $k > r$

$$\phi(\sigma_1\sigma_2\cdots\sigma_k) \leq M^r \cdot (\hat{\phi} - \alpha)^w, \tag{3}$$

where $w$ is an integer in $(k-r,k]$ and

$$M = \max(1, \phi(i);\ i = 1, 2, \ldots, N). \tag{4}$$

(Indeed, in this case the prefix $\sigma_1\sigma_2\cdots\sigma_k$ is divided into the concatenation of words of length at most $r$, each belonging to $\mathcal{C}$, and a "remaining" part shorter than $r$. Applying (1), (2) and that $\phi(\sigma_1\sigma_2\cdots\sigma_m) \le M^m$ for any $\sigma_1\sigma_2\cdots\sigma_m$, we obtain (4).)

*Step* 3. It follows from (3) by taking max. for all prefixes in $\mathcal{J}$ of length $k$ that also

$$\hat{\phi}_k \le M^r \cdot (\hat{\phi} - \sigma)^w$$

and hence

$$\hat{\phi} = \limsup\{\hat{\phi}_k^{\frac{1}{k}};\ k = 1, 2, \ldots\} \le \lim_{k\to\infty} M^{\frac{r}{k}} \cdot \lim_{k\to\infty} (\hat{\phi} - \alpha)^{\frac{w}{k}} = \hat{\phi} - \alpha.$$

Thus a contradiction is obtained and hence $\mathcal{T}$ is infinite.

We shall formulate Proposition 1 using concepts from graph theory. Combining the graph theoretical formulation of Proposition 1 with the celebrated König Lemma for infinite trees, a weighted version of the König Lemma will be obtained.

Consider the *infinite graph* $\mathcal{G}$ with vertex set $\{\sigma_1\sigma_2\cdots\sigma_n;\ n = 1, 2, \ldots\}$ and edges $[\sigma_1\sigma_2\cdots\sigma_k, \sigma_1\sigma_2\cdots\sigma_{k+1}]$ for $k = 1, 2, \ldots$.

If we add the symbol $\varnothing$ as a new vertex and $[\varnothing, \sigma_i]$ for $i = 1, 2, \ldots, N$ as new edges, then $\mathcal{G}$ is a *rooted tree with root* $\varnothing$. In fact, $\mathcal{G}$ is a weighted tree with weights $\phi = \phi(\sigma_1\sigma_2\cdots\sigma_k)$ for the vertex $\sigma_1\sigma_2\cdots\sigma_k$.

In this setting $\sigma \in \mathcal{J}$ is considered as an infinite path of $\mathcal{G}$.

More precisely, considering the mapping $\Gamma$ which maps each $\sigma \in \mathcal{J}$ to the infinite path with vertices $\{\sigma_1\sigma_2\cdots\sigma_n;\ n = 1, 2, \ldots\}$, $\Gamma$ is a 1-1 mapping. $\Gamma$ is NOT necessarily ONTO.

**Example.** Let $\sigma = 100110001110000\ldots$, i.e. the infinite string $\sigma$ consists of blocks of concatenation of $k$ zeros and $k$ ones for $k = 2, 3, \ldots$. Now if $\mathcal{J} = \{S^k\sigma;\ k = 0, 1, \ldots\}$, then $\Gamma$ is not onto since the infinite path with vertices $\{\sigma_1\sigma_2\cdots\sigma_n;\ n = 1, 2, \ldots\}$ such that $\sigma_i = 1$ for every symbol has no counterpart in $\mathcal{J}$ (see Figure 1).

$\mathcal{J}$ is called *complete* if $\Gamma$ is onto. *From now on we shall suppose that* $\mathcal{J}$ *is shift-invariant and complete.*

Now the *Weighted König Lemma* is the following.

**Proposition 2.** *There exists* $\sigma \in \mathcal{J}$ *such that*

$$\phi(\sigma_1\sigma_2\cdots\sigma_k) \ge \hat{\phi}^k, \qquad k = 1, 2, \ldots.$$

*In other words, there is an infinite path in* $\mathcal{G}$ *with weights not less than* $\hat{\phi}^k$.

*Proof.* It is easy to verify that the subgraph of $\mathcal{G}$ corresponding to $\mathcal{T}$ is also a rooted tree with root $\varnothing$. By Proposition 1, $\mathcal{T}$ is an infinite tree.

The König Lemma says, that in an infinite rooted tree, with all vertices of finite degree, there is an infinite path starting from the root. Apply the König Lemma to the subgraph $\mathcal{T}$.

In this paper we shall apply this Weighted König Lemma for the case when

$$\phi(\sigma_1\sigma_2\cdots\sigma_n) = \|A_{\sigma n}\cdots A_{\sigma 2}A_{\sigma 1}\|.$$

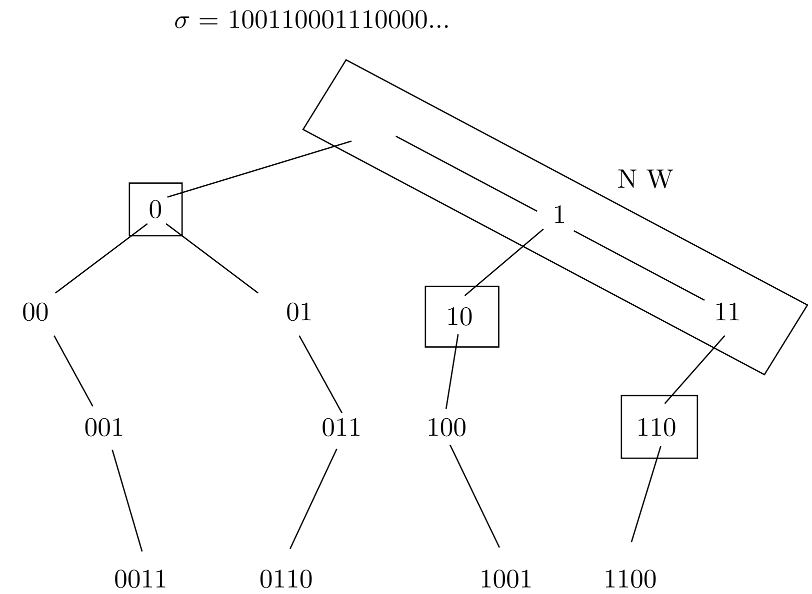


FIGURE 1

## 4. THE CONVERGENCE OF THE INFINITE PRODUCT OF OPERATORS $\{A_k;\ k = 1, 2, \dots, N\}$

Now we shall turn to our main subject, and infinite product $\cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}$ ($\sigma_i \in \{1, 2, \dots, N\}$) of linear operators of a Hilbert space.

The set $\mathcal{S}$ of infinite products of operators $\{A_i;\ i = 1, 2, \dots, N\}$ is called *shift-invariant*, resp. *complete*, if the corresponding $\mathcal{J}$ is shift-invariant, resp. complete. We shall suppose that $\mathcal{S}$ is shift-invariant and complete.

The key object in our investigations is $\hat{\phi}$, defined in Proposition 1. If

$$\phi(\sigma_1 \sigma_2 \cdots \sigma_n) = \|A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}\|,$$

then $\hat{\phi}$ is called *the joint spectral radius* $\hat{\rho}(\mathcal{S})$.

**Theorem 1.** *$\hat{\rho}(\mathcal{S}) < 1$ if and only if every product in $\mathcal{S}$ tends to zero.*

*More particularly, if $\hat{\rho}(\mathcal{S}) < 1$, then every product in $\mathcal{S}$ tends to zero according to* (5) *at an exponential rate.*

*Proof.* If $\hat{\rho}(\mathcal{S}) < 1$, then there exists $r$ with $\rho_r^{1/r} < 1$. Since

$$\|A_{\sigma r} \cdots A_{\sigma 2} A_{\sigma 1}\| < \rho_r \quad \text{for every } \sigma \in \mathcal{J}$$

by definition, it follows for $k > r$ that

$$\|A_{\sigma k} \cdots A_{\sigma 2} A_{\sigma 1}\| < M^r \cdot \rho_r^w \qquad (0 < \rho < 1) \tag{5}$$

where $M = \max\{1, \|A_1\|;\ i = 1, 2, \dots, N\}$ and $w$ is the integer part of $k/r$.

It follows from (5) that

$$\cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} \Rightarrow 0$$

at an exponential rate.

If $\hat{\rho}(\mathcal{S}) \geq 1$, then by the Weighted König Lemma there is an $\omega \in \mathcal{S}$ such that $\|A_{\omega k} \cdots A_{\omega 2} A_{\omega 1}\| \geq 1$ for every $k$ and hence the infinite product $\cdots A_{\omega k} \cdots A_{\omega 2} A_{\omega 1}$ does not tend to zero.

We shall show that if there is an orthogonal decomposition $\mathcal{H} = F \oplus V$ such that $\hat{\rho} < 1$ on $V$ and every $u \in F$ is a fixed point of every $A_i$, then every infinite product $M(\sigma) := \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}$ for $\sigma \in \mathcal{J}$ is convergent. More precisely, we have

**Theorem 2.** *Let $\mathcal{S}_V$ be the restriction of $\mathcal{S}$ to $V$. I.e. the elements of $\mathcal{S}_V$ are infinite products of $\{\mathfrak{U}_i;\ i = 1, 2, \ldots, N\}$ and*

$$\cdots \mathfrak{U}_{\sigma n} \cdots \mathfrak{U}_{\sigma 2} \mathfrak{U}_{\sigma 1} \in \mathcal{S}_V \quad \textit{iff} \quad \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} \in \mathcal{S}.$$

*If we have an orthogonal decomposition*

$$\mathcal{H} = F \oplus V$$

*with*

$$F = \{u : A_{\sigma i} u = u;\ i = 1, 2, \ldots\} \quad \textit{and} \quad \hat{\rho}(\mathcal{S}_V) < 1,$$

*then $M(\sigma) = \cdots A_{\sigma k} \cdots A_{\sigma 2} A_{\sigma 1}$ is convergent for every $\sigma \in \mathcal{J}$. Moreover, identifying the operator and its* (OH) *decomposition one has*

$$M(\sigma) := \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} = \begin{pmatrix} P_F & \mathfrak{S} \\ 0 & 0 \end{pmatrix}$$

*where*

$$\mathfrak{S} = \mathfrak{C}_{\sigma 1} + \mathfrak{C}_{\sigma 2} \mathfrak{U}_{\sigma 1} + \cdots + \mathfrak{C}_{\sigma n} \mathfrak{U}_{\sigma [n-1]} \cdots \mathfrak{U}_{\sigma 1} + \cdots .$$

*Proof.* The (OH) decomposition of $A_i$ corresponding to $\mathcal{H} = F \oplus V$ is

$$\text{(M)} \qquad \begin{pmatrix} P_F & \mathfrak{C}_i \\ 0 & \mathfrak{U}_i \end{pmatrix},$$

since in this case $P_F A P_F = P_F$ and $P_V A P_F = 0$. It follows from (M) by induction that

$$A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} = \begin{pmatrix} P_F & \mathfrak{C}_{\sigma 1} + \mathfrak{C}_{\sigma 2} \mathfrak{U}_{\sigma 1} + \cdots + \mathfrak{C}_{\sigma n} \mathfrak{U}_{\sigma [n-1]} \cdots \mathfrak{U} \sigma_1 \\ 0 & \mathfrak{U}_{\sigma n} \cdots \mathfrak{U}_{\sigma 2} \mathfrak{U}_{\sigma 1} \end{pmatrix}.$$

Since $\hat{\rho}(\mathcal{S}_V) < 1$ by Theorem 1, we have that for every $\sigma \in \mathcal{J}$ and $k > r$

$$\text{(5}^*\text{)} \qquad \|\mathfrak{U}_{\sigma k} \cdots \mathfrak{U}_{\sigma 2} \mathfrak{U}_{\sigma 1}\| < M^r \cdot \rho_r^w \qquad (0 < \rho < 1)$$

where $M = \max\{1, \|\mathfrak{U}_i\|;\ i = 1, 2, \ldots, N\}$ and $w$ is the integer part of $k/r$.

It follows from ($5^*$) that the infinite product $\cdots \mathfrak{U}_{\sigma n} \cdots \mathfrak{U}_{\sigma 2} \mathfrak{U}_{\sigma 1}$ tends to zero at an exponential rate and hence the series

$$(\Sigma) \qquad \mathfrak{C}_{\sigma 1} + \mathfrak{C}_{\sigma 2} \mathfrak{U}_{\sigma 1} + \cdots + \mathfrak{C}_{\sigma n} \mathfrak{U}_{\sigma [n-1]} \cdots \mathfrak{U}_{\sigma 1} + \cdots$$

is also convergent.

The next theorem is the converse of Theorem 2 in a certain sense. It roughly says that if the infinite product $M(\sigma)$ is convergent for every $\sigma \in \mathcal{J}$, then an orthogonal decomposition like in Theorem 2 must exist.

*Let $\sigma^{[\infty]}$ be the set of those symbols which occur infinitely many times in $\sigma$.* It is obvious that for every infinite string $\sigma$ of $N$ symbols there is a $K$ such that every symbol $\sigma_i$ of $S^K \sigma$ belongs to $\sigma^{[\infty]}$.

We emphasize that $\sigma^{[\infty]}$ depends on $\sigma$. If $\sigma$ and $\omega$ are different strings of $N$ symbols, both belonging to $\mathcal{J}$, then $\sigma^{[\infty]}$ and $\omega^{[\infty]}$ are different, in general.

**Theorem 3.** *If*

I. $M(\sigma) := \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}$ *is convergent for every* $\sigma \in \mathcal{J}$;

II. $\sigma \Rightarrow M(\sigma)$ *is continuous (for* $\sigma \in \mathcal{J}$*)*;

III. *For* $\sigma, \omega \in \mathcal{J}$ *and for every* $K$

$$\sigma_1 \sigma_2 \cdots \sigma_K \omega_1 \omega_2 \cdots \in \mathcal{J};$$

IV. *There exists a* $j$ *such that* $(A_j^n; n = 1, 2, \dots\}$ *is convergent and* $j \in \sigma^{[\infty]}$ *for every* $\sigma \in \mathcal{J}$,

*then*

$$\mathcal{H} = F \oplus V$$

*where*

$$F = \{u : A_i u = u; i \in \sigma^{[\infty]}\} \quad \textit{and} \quad P_V(A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}) P_V \Rightarrow 0 \quad \textit{for every } \sigma \in \mathcal{J}.$$

*Proof. Step 1.* If

$$\lim_{n \to \infty} A_j^n = M(j),$$

then

$$F(A_j) \subseteq F(M(j)) \subseteq \operatorname{range} M(j).$$

Indeed, if $A_j v = v$, then $A_j^n v = v$ for every $n$ and hence $M(j)v = v$ since $\{A_j^n; n = 1, 2, \dots\}$ is convergent. The second inclusion is obvious.

*Step* 2.

$$\operatorname{range} M(\sigma) \subseteq \bigcap \{F(A_i); i \in \sigma^{[\infty]}\}.$$

In fact, the prefixes of $\sigma$ ending with $i \in \sigma^{[\infty]}$ are a subsequence of $\{\sigma_1 \sigma_2 \cdots \sigma_n; n = 1, 2, \dots\}$ and hence, the corresponding products of operators are a subsequence of $\{A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1}; n = 1, 2, \dots\}$. Moreover,

$$\lim A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} = \lim A_i A_{\sigma k_i} \cdots A_{\sigma 2} A_{\sigma 1} = A_i \lim A_{\sigma k_i} \cdots A_{\sigma 2} A_{\sigma 1}.$$

It follows that

$$\text{(A)} \qquad A_i M(\sigma) = M(\sigma) \quad \text{for } i \in \sigma^{[\infty]},$$

i.e.

$$A_i M(\sigma) v = M(\sigma) v \quad \text{for } i \in \sigma^{[\infty]},\ v \in \mathcal{H}.$$

*Step* 3. Let $\sigma \in \mathcal{J}$ and $K$ such that $\sigma i \in \sigma^{[\infty]}$ for $i > K$. Let

$$\sigma_{[m]} = jj \cdots \overset{m}{\overline{j}} \sigma_K \sigma_{K+1} \cdots \sigma_n \cdots ;$$

then $\sigma_{[m]} \Rightarrow jj \cdots j \cdots$ and hence $M(\sigma_{[m]}) \Rightarrow M(j)$. Moreover, $\sigma_{[m]}^{[\infty]} = \sigma^{[\infty]}$.

*Step* 4. Summarizing, we obtain

$$\text{(6)} \qquad \operatorname{range} M(\sigma_{[m]}) \subseteq \bigcap \{F(A_i); i \in \sigma^{[\infty]}\} \subseteq F(A_j) \subseteq F(M(j)) \subseteq \operatorname{range} M(j).$$

Since $M(\sigma_{[m]}) \Rightarrow M(j)$, it follows that $M(\sigma_{[m]})v \Rightarrow M(j)v$ for every $v \in \mathcal{H}$ and hence $\operatorname{range} M(j)$ is included in the closure of $\{\operatorname{range} M(\sigma_{[m]}); m = 1, 2, \dots\}$. Hence it follows from (6) that

$$\text{(7)} \qquad \bigcap \{F(A_i);\ i \in \sigma^{[\infty]}\} = F(A_j)$$

since $F(A_i)$ is closed.

*Step* 5. It follows from (7) that $F(A_j) \subseteq F(A_i)$ for every $i \in \sigma^{[\infty]}$, i.e. if $u \in F(A_j)$, then $A_i u = u$ for every $i \in \sigma^{[\infty]}$. It follows that $P_F A_i P_F = P_F$ and $P_V A_i P_F = 0$ in the decomposition (OH) when $P_F$ is the projection onto $F(A_j)$ and hence

$$A_i = \begin{pmatrix} P_F & \mathfrak{C}_i \\ 0 & \mathfrak{U}_i \end{pmatrix} \quad \text{for } i \in \sigma^{[\infty]}.$$

It follows from Lemma 1 and Lemma 3 that $\mathfrak{U}_j^n \Rightarrow 0$, moreover, from (A) follows that also

$$\text{(A}^*\text{)} \qquad \mathfrak{U}_j^n \cdot \mathfrak{M}(\sigma) = \mathfrak{M}(\sigma) \quad \text{since } j \in \sigma^{[\infty]}$$

and hence $\mathfrak{M}(\sigma) := \cdots \mathfrak{U}_{\sigma n} \cdots \mathfrak{U}_{\sigma[K-1]} \mathfrak{U}_{\sigma_K}$ tends to zero.

**Corollary.** *It follows from the foregoing also that*

$$M(\sigma) := \cdots A_{\sigma n} \cdots A_{\sigma 2} A_{\sigma 1} = \begin{pmatrix} P_F & \mathfrak{C} \\ 0 & 0 \end{pmatrix} \quad \textit{if } \sigma i \in \sigma^{[\infty]}$$

*where*

$$(\Sigma) \qquad \mathfrak{S} = \mathfrak{C}_{\sigma 1} + \mathfrak{C}_{\sigma 2} \mathfrak{U}_{\sigma 1} + \cdots + \mathfrak{C}_{\sigma n} \mathfrak{U}_{\sigma[n-1]} \cdots \mathfrak{U}_{\sigma 1} + \cdots .$$

*Remark.* It is sufficient in Theorem 2 that

$$F = \{u : A_i u = u;\ i \in \sigma^{[\infty]}\}.$$

## 5. Conclusion

It was shown that the main results of [1] remain valid for a complete, shift-invariant subset $\mathcal{J}$ of all infinite products of a finite set $\{A_i;\ i = 1, 2, \ldots, N\}$ of bounded linear operators of a Hilbert space $\mathcal{H}$.

Our main observation is that $\mathcal{H}$ is decomposed into the direct sum of subspaces $F$ and $V$ such that for every infinite product $M(\sigma) \in \mathcal{S}$,

$$M(\sigma)u = u \quad \text{for} \quad u \in F, \qquad M(\sigma)v = 0 \quad \text{for } v \in V$$

assuming that $M(\sigma) \in B(\mathcal{H})$. Moreover,

$$\hat{\rho}(\mathcal{S}) < 1 \quad \text{iff} \quad M(\sigma) = 0.$$

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