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AVERAGES OF OPERATORS AND THEIR POSITIVITY

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Abstract. Let T be a bounded linear operator on a Hilbert space. We prove that T is positive, if there exists a positive integer N such that

$$\|I - \frac{1}{N+1}\sum_{i=k}^{k+N} T^i\|, \|I - \frac{1}{N+2}\sum_{i=k}^{k+N+1} T^i\| \le 1$$

for any non-negative integer k. For several commuting operators, we can extend this result and get the similar statement.

1. Introduction

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. If $\|I - T^n\| \le \delta < 1$ for all $n \in \mathbb{N}$, then $T = I$. This fact is well-known as the theorem of Cox ([1],[6]). For several power bounded operators $T_1, T_2, \dots, T_n$, the authors show the following statement as an extension of the above one:

$$\|I - \frac{1}{|d(n,k)|}\sum_{\overrightarrow{d} \in d(n,k)} T_1^{d_1} T_2^{d_2} \cdots T_n^{d_n}\| \le \delta < 1 \text{ implies } T_1 = T_2 = \cdots = T_n = I,$$

where $d(n,k) = \{\overrightarrow{d} = (d_1, d_2, \dots, d_n) | d_i \ge 0, \sum_{i=1}^n d_i = k\}$ $(k \ge 0)$ and $|d(n,k)|$ is the cardinal number of the set $d(n,k)$.

In the above statement, the distance between averages and the identity is less than 1. In this paper we treat the case that the distance is less than or equal to 1, and we get the following result:

$$(*)\ 0 \le T \le I \text{ if } \|I - T^n\| \le 1 \text{ for all } n \in \mathbb{N},$$

$$(**)\ 0 \le T_i^N \le I \text{ if } \|I - \frac{1}{|d(n,k)|}\sum_{\overrightarrow{d} \in d(n,k)} T_1^{d_1+e_1} T_2^{d_2+e_2} \cdots T_n^{d_n+e_n}\| \le 1$$

for all $k \ge N$, $e_1, e_2, \dots, e_n \in \mathbb{N} \cup \{0\}$, where $T_1, T_2, \dots, T_n$ are commuting operators (i.e., $T_i T_j = T_j T_i$).

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2. Positive contractions

For a bounded linear operator T on $\mathcal{H}$, we define the numerical range $W(T)$ of T by

$$W(T) = \{(Tx|x)|\ x \in \mathcal{H},\ \|x\| = 1\}.$$

T. Kato [4] shows that $W(f(T))$ is contained in the closed convex hull of $f(\Pi)$ whenever f is holomorphic on a neighborhood of the closed right half plane Π of $\mathbb{C}$ and $W(T)$ is contained in Π. Using this fact, C. R. de Prima and B. K. Richard [2] show that T is positive whenever $W(T^n)$ is contained in Π for all $n \in \mathbb{N}$. Concerning the statement $(*)$ we prove the following theorem.

Theorem 1. *Let T be an operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(1) $0 \leq T \leq I$.
(2) $\|I - T^n\| \leq 1$ *for all* $n \in \mathbb{N}$.
(3) *There exists a positive integer N such that*

$$\|I - \frac{1}{n+1}\sum_{j=k}^{k+n} T^j\| \leq 1$$

for all $n \geq N$ and $k \in \mathbb{N} \cup \{0\}$.
(4) *There exists a positive integer N such that*

$$\|I - \frac{1}{n+1}\sum_{j=k}^{k+n} T^j\| \leq 1$$

for $n \in \{N, N+1\}$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are immediate.

(2)⇒(1) For $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$|1 - (T^n x|x)| = |((I - T^n)x|x)| \leq \|I - T^n\| \leq 1,$$

so $W(T^n)$ is contained in Π for all $n \in \mathbb{N}$. This means that T is positive. If there exists a positive number $t \in Sp(T)$ such that $t > 1$, then there exists a positive integer n such that $t^n > 2$. So we have that the spectral radius of $I - T^n$ is more than 1. This contradicts $\|I - T^n\| \leq 1$. Thus we have $0 \leq T \leq I$.

(4)⇒(1) We define $T(n, k)$ by

$$T^k(\frac{1}{n+1}\sum_{j=0}^{n} T^j) \ \text{ for any } n \in \mathbb{N} \text{ and } k \in \mathbb{N} \cup \{0\}.$$

For any $m \geq 2$, $T(n, k)^m$ is represented as a convex combination of elements in $\{T(n, l)|l \in \mathbb{N} \cup \{0\}\}$. In fact,

$$T(n, k)^m = \frac{1}{(n+1)^{m-1}} \sum_{j_1, j_2, \ldots, j_{m-1}=0}^{n} T(n, mk + j_1 + \cdots + j_{m-1}).$$

So we have $\|I - T(N, k)^m\| \leq 1$ and $\|I - T(N+1, k)^m\| \leq 1$. By the implication (2)⇒(1), $T(N, k)$ and $T(N+1, k)$ are positive. Since $T = (N+2)T(N+1, 1) -$

$(N+1)T(N,2)$, T is self-adjoint. For a real number r, the number r belongs to the closed interval $[0,1]$, if r satisfies

$$|1 - r^k \frac{1}{n+1}\sum_{j=0}^{n} r^j| \leq 1 \quad \text{for all } n = N, N+1 \text{ and } k \geq 0.$$

Therefore $0 \leq T \leq I$. □

Remark. The implication (2)⇒(1) is suggested by Y. Nakamura. In relation to the statement of the above Theorem, we also note the following fact. Let $\omega = \frac{-1+\sqrt{3}i}{2}$, and set $T = \omega I$. By simple calculation, we have

$$\|I - \frac{1}{3}\sum_{j=k}^{k+2} T^j\| = 1 \quad \text{for all } k \in \mathbb{N} \cup \{0\}$$

$$\|I - \frac{1}{n+1}\sum_{j=0}^{n} T^j\| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Then T is not positive, nor self-adjoint.

3. Nilpotent operators and positive contractions

Proposition 2. *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ and n a positive integer. Then the following are equivalent:*

(1) *T^n and T^{n+1} are positive contractions.*
(2) *T is a direct sum of operators N and S such that*

$$N^n = 0, \qquad 0 \leq S \leq I.$$

(3) *For any integer $m \geq n$, $\|I - T^m\| \leq 1$.*

Proof. The implication (2)⇒(3) is trivial, and the implication (3)⇒(1) follows from Theorem 1. So it suffices to show (1)⇒(2). At first we remark that $KerT^n = KerT^{n+1}$, because

$$KerT^n \subset KerT^{n+1} \subset \cdots \subset KerT^{2n} = Ker(T^n)^*T^n = KerT^n.$$

We decompose the Hilbert space $\mathcal{H}$ into the direct sum of $KerT^n$ and $(KerT^n)^\perp = \overline{Range(T^n)}$. Then T, T^n and T^{n+1} have the form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad T^n = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}, \quad T^{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$$

with respect to the decomposition $(KerT^n) \oplus (KerT^n)^\perp$. By the calculation

$$T^{n+1} = TT^n = \begin{pmatrix} 0 & BE \\ 0 & DE \end{pmatrix}$$
$$= T^nT = \begin{pmatrix} 0 & 0 \\ EC & ED \end{pmatrix},$$

we have $BE = 0$, $EC = 0$ and $DE = ED = F = F^* = (ED)^* = D^*E$. Since $\overline{Range(E)} = (KerE)^\perp = (KerT^n)^\perp$, $B = 0$, $C = 0$ and $D = D^*$. In particular, we have $A^n = 0$. By the positivity and contactivity of T^n and T^{n+1}, we can get that D is positive and contractive. □

The following statement is a direct result of the above proof.

Corollary 3. *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ and n a positive integer. Then the following are equivalent:*
(1) *T^n and T^{n+1} are self-adjoint.*
(2) *T is a direct sum of operators N and S such that*

$$N^n = 0, \qquad S = S^*.$$

Let $T_1, T_2, \dots, T_n$ be bounded linear operators on a Hilbert space $\mathcal{H}$. In the rest of the paper, we assume that operators $T_1, T_2, \dots, T_n$ are commuting with each other. We define the set $d(n,k)$ by

$$d(n,k) = \{\overrightarrow{d} = (d_1, d_2, \cdots, d_n) | d_i \in \mathbb{N} \cup \{0\}, \sum_{i=1}^{n} d_i = k\} \quad (k \in \mathbb{N} \cup \{0\}),$$

and the partial average $R(n,k)$ for $T_1, T_2, \dots, T_n$ by

$$R(n,k) = \frac{1}{|d(n,k)|} \sum_{\overrightarrow{d} \in d(n,k)} T_1^{d_1} T_2^{d_2} \dots T_n^{d_n},$$

where $|d(n,k)|$ denotes the cardinal number of the set $d(n,k)$ (see [5]). Furthermore, for any $\overrightarrow{e} = (e_1, e_2, \cdots, e_n) \in (\mathbb{N} \cup \{0\})^n$, we define the $\overrightarrow{e}$-shifted partial average $R(n,k,\overrightarrow{e})$ by

$$R(n,k,\overrightarrow{e}) = \frac{1}{|d(n,k)|} \sum_{\overrightarrow{d} \in d(n,k)} T_1^{d_1+e_1} T_2^{d_2+e_2} \cdots T_n^{d_n+e_n}.$$

By using this notation, we can get the following result concerning the statement $(**)$.

Theorem 4. *Let $T_1, T_2, \dots, T_n \in B(\mathcal{H})$ be commuting with each other. Then the following are equivalent:*
(1) *There exists a positive integer K such that*

$$\sup_{\overrightarrow{e}} \|I - R(n,k,\overrightarrow{e})\| \leq 1 \quad for\ all\ k \geq K.$$

(2) *There exists a positive integer K such that*

$$\sup_{\overrightarrow{e}} \{\|I - R(n,K,\overrightarrow{e})\|, \|I - R(n,K+1,\overrightarrow{e})\|\} \leq 1.$$

(3) *There exist a closed subspace $\mathcal{K}$, pairwise commuting nilpotent operators $\{N_i\}$ acting on $\mathcal{K}$ and pairwise commuting positive contractions $\{S_i\}$ acting on $\mathcal{K}^\perp$ such that $T_i = N_i + S_i$ for all $i = 1, 2, \dots, n$.*

Proof. (1)⇒(2) It is trivial.

(3)⇒(1) We assume $N_i^{n_i} = 0$. Then for every $k > \sum_{i=1}^{n} n_i$ we have

$$\begin{aligned} R(n,k,\overrightarrow{e}) &= \frac{1}{|d(n,k)|} \sum_{\overrightarrow{d} \in d(n,k)} T_1^{d_1+e_1} T_2^{d_2+e_2} \cdots T_n^{d_n+e_n} \\ &= \frac{1}{|d(n,k)|} \sum_{\overrightarrow{d} \in d(n,k)} S_1^{d_1+e_1} S_2^{d_2+e_2} \cdots S_n^{d_n+e_n}. \end{aligned}$$

Therefore $R(n,k,\overrightarrow{e})$ is a positive contraction, which shows $\|I - R(n,k,\overrightarrow{e})\| \leq 1$.

(2)⇒(3) Let k be equal to K or $K+1$. Since

$$R(n,k,\overrightarrow{e})R(n,l,\overrightarrow{f}) = \frac{1}{|d(n,l)|}\sum_{\overrightarrow{d}\in d(n,l)} R(n,k,\overrightarrow{e}+\overrightarrow{f}+\overrightarrow{d})$$

and $\|I - R(n,k,\cdot)\| \leq 1$, we have

$$\sup_{m\in\mathbb{N}} \|I - R(n,k,\overrightarrow{e})^m\| \leq 1$$

for all $\overrightarrow{e} \in (\mathbb{N}\cup\{0\})^n$. By Theorem 1, we have $0 \leq R(n,k,\overrightarrow{e}) \leq 1$. We put

$$\overrightarrow{e_j} = (0,0,\cdots,0,\overset{j}{\overbrace{1}},0,\cdots,0).$$

The identities

$$R(n,K+2,\overrightarrow{0}) = \frac{1}{|d(n,K+2)|}\sum_{i=1}^{n} |d(n,K+1)|R(n,K+1,\overrightarrow{e_i})$$

and

$$T_i^{k+1} = |d(n,k+1)|R(n,k+1,\overrightarrow{0}) - \sum_{j\neq i} |d(n,k)|R(n,k,\overrightarrow{e_j})$$

imply that T_i^{k+1} is self-adjoint for $k = K,\ K+1$ and $i = 1,2,\ldots,n$. By Corollary 3, T_i is represented as the direct sum of a nilpotent operator N_i such that $N_i^{K+1} = 0$ and a self-adjoint operator S_i. Since $T_iT_j = T_jT_i$, we have

$$S_iS_j = S_jS_i,\ S_iN_j = N_jS_i.$$

Let Q_i be the orthogonal projection onto $KerT_i^{k+1}$ and let $P_i = I - Q_i$. It follows from the property $T_iT_j = T_jT_i$ that

$$Q_iQ_j = Q_jQ_i,\ KerS_i = Q_i\mathcal{H}.$$

If we put $\mathcal{K} = Q_1Q_2\cdots Q_n\mathcal{H}$, then $\mathcal{K}^{\perp} = (I-Q_1)\vee(I-Q_2)\vee\cdots\vee(I-Q_n)\mathcal{H} = (I - Q_1Q_2\cdots Q_n)\mathcal{H}$.

Let us prove $N_i(I - Q_1Q_2\cdots Q_n) = 0$ for all i. It is sufficient to prove that $N_n(I - Q_1Q_2\cdots Q_n) = 0$. We put

$$l = \max\{m|\ N_n^m(I - Q_1Q_2\cdots Q_n) \neq 0\},$$
$$\mathcal{L} = KerN_n^l(I - Q_1Q_2\cdots Q_n).$$

We assume $l \geq 1$. Then we can choose a unit vector $x \in \mathcal{L}^{\perp}$ such that $x = (I - Q_1Q_2\cdots Q_n)x$, $0 \neq N_n^l x \in KerT_n \subset \mathcal{L}$. Since

$$\begin{aligned}&\|(I - R(n,k,\overrightarrow{e}+l\overrightarrow{e_n}))x\|^2\\ &= \|x\|^2 - 2Re\langle R(n,k,\overrightarrow{e}+l\overrightarrow{e_n})x \mid x\rangle + \|R(n,k,\overrightarrow{e}+l\overrightarrow{e_n})x\|^2\\ &= 1 + \|R(n,k,\overrightarrow{e}+l\overrightarrow{e_n})x\|^2\end{aligned}$$

and the assumption $\|I-R(n,k,\overrightarrow{e}+l\overrightarrow{e_n})\|\leq 1$, we have $R(n,k,\overrightarrow{e}+l\overrightarrow{e_n})x=0$,$k=K,\ K+1$. It follows from $x=(I-Q_1Q_2\cdots Q_n)x$ that, for each $i=1,2,\dots n-1$,

$$\begin{aligned}T_i^{K+1}N_n^l x &= |d(n,K+1)|R(n,K+1,\overrightarrow{0})N_n^l x\\ &\qquad -\sum_{j\neq i}|d(n,K)|R(n,K,\overrightarrow{e_j})N_n^l x\\ &= |d(n,K+1)|R(n,K+1,l\overrightarrow{e_n})x\\ &\qquad -\sum_{j\neq i}|d(n,K)|R(n,K+1,\overrightarrow{e_j}+l\overrightarrow{e_n})x\\ &= 0.\end{aligned}$$

Therefore $P_iN_n^lx=0$. Consequently $x\in KerN_n^l(P_1\vee P_2\vee\cdots\vee P_n)=\mathcal{L}$. This contradicts the assumption $l\geq 1$ So we get $N_n(I-Q_1Q_2\cdots Q_n)=0$. Therefore we have $N_i\in B(Q_1\wedge Q_2\wedge\cdots\wedge Q_nH)$ and $S_i\in B(P_1\vee P_2\vee\cdots\vee P_nH)$ for all $i=1,2,\dots,n$.

It remains to prove $S_i\geq 0$. By the Gelfand-Naimark theorem, the C*-algebra generated by $\{S_i\}$ can be identified with the algebra of all the continuous functions on a compact Hausdorff space Ω. In this identification, we can see each S_i is a real-valued continuous function on Ω. We put $\mathcal{E}=\{(f_1,f_2,\cdots,f_n)\in(\mathbb{N}\cup\{0\})^n \mid f_i\geq K, i=1,2,\dots,n\}$. We can get the positivity of $S_1,S_2,\cdots,S_n$ by the relations

$$R(n,k,\overrightarrow{f})=\frac{1}{|d(n,k)|}\sum_{\overrightarrow{d}\in d(n,k)}S_1^{d_1+f_1}S_2^{d_2+f_2}\cdots S_n^{d_n+f_n},$$

$$S_i^{K+1+f_i}=|d(n,K+1)|R(n,K+1,\overrightarrow{f})-|d(n,K)|\sum_{j\neq i}R(n,K,\overrightarrow{f}+\overrightarrow{e_j})$$

and

$$0\leq R(n,k,\overrightarrow{f})$$

for all $\overrightarrow{f}\in\mathcal{E}$, $k=K,\ K+1$. This completes the proof. □

Remark. If we consider the case $T=T_1=T_2=\cdots=T_n$, then the statement of Theorem 4 implies that of Theorem 1. Without the assumption that $T_1,T_2,\cdots,T_n$ are pairwise commuting, we cannot show the implication (2)⇒(3). For example,

$$T_1=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ T_2=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

are not commuting and the above implication (2)⇒(3) does not hold for T_1,T_2.

4. Application of Theorem 1

Let A be a unital involutive Banach algebra over $\mathbb{C}$. We call A an A*-algebra if A has a faithful *-representation on a Hilbert space.

Theorem 5. *Let A be a unital A*-algebra. If S is a semi-group of A contained in $\{a\in A \| \|a-I\|\leq 1\}$, then S is abelian.*

Proof. Let π be a unital, faithful *-representaion of A. Then we have

$$\begin{aligned}\|\pi(x)\|^2 &= \|\pi(x)^*\pi(x)\| = \|\pi(x^*x)\| \\ &= \sup\{|\lambda| | \lambda \in Sp(\pi(x^*x))\} \leq \sup\{|\lambda| | \lambda \in Sp(x^*x)\} \\ &= \lim_{n\to\infty} \|(x^*x)^n\|^{1/n} \leq \|x^*x\| \leq \|x\|^2.\end{aligned}$$

Therefore the semi-group $\pi(S)$ is contained in $\{x \in \pi(A) | \|x - I\| \leq 1\}$. For $a \in S$, $\|\pi(a)^n - I\| \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Using Theorem 1, we have $\pi(a), \pi(b)$ and $\pi(ab)$ are positive contractions for $a, b \in S$. Since π is faithful and the calculation,

$$\pi(a)\pi(b) = \pi(ab) = \pi(ab)^* = \pi(b)^*\pi(a)^* = \pi(b)\pi(a),$$

we have that S is abelian. □

If A is a unital involutive Banach algebra and not an A^*-algebra, then the above theorem is not necessarily valid. In fact, we have the following example:

Example. For an element (α, β) of $\mathbb{C}^2$, we consider the following norms:

$$\|(\alpha, \beta)\|_1 = |\alpha| + |\beta|, \quad \|(\alpha, \beta)\|_\infty = \max\{|\alpha|, |\beta|\}.$$

Let x be an element of $M_2(\mathbb{C})$. We can regard x as a bounded linear map from $(\mathbb{C}^2, \|\cdot\|_1)$ (resp. $(\mathbb{C}^2, \|\cdot\|_\infty)$) to $(\mathbb{C}^2, \|\cdot\|_1)$ (resp. $(\mathbb{C}^2, \|\cdot\|_\infty)$), and consider the following norms:

$$\|x\|_1 = \max\{|a| + |c|, |b| + |d|\} \quad (x \in B(\mathbb{C}^2, \|\cdot\|_1)),$$

$$\|x\|_\infty = \max\{|a| + |b|, |c| + |d|\} \quad (x \in B(\mathbb{C}^2, \|\cdot\|_\infty)),$$

where x has the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define the conjugate linear map s from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ by

$$s(x) = s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}.$$

Then we can easily check $\|s(x)\|_\infty = \|x\|_1$ and $\|s(x)\|_1 = \|x\|_\infty$. We consider the algebra $A = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and define the norm and the involution on A by

$$\|(x, y)\| = \max\{\|x\|_1, \|y\|_\infty\} \text{ and } (x, y)^* = (s(y), s(x)),$$

where $x, y \in M_2(\mathbb{C})$. Then A becomes an involutive Banach algebra. We set S to be the semi-group of A generated by $(x_1, 0)$ and $(x_2, 0)$, where $x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. In fact, S consists of only two elements $(x_1, 0), (x_2, 0)$. Clearly S is not abelian and

$$\|(I, I) - (x_1, 0)\|, \|(I, I) - (x_2, 0)\| \leq 1.$$

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