

ON THE TAYLOR SPECTRUM OF LEFT-RIGHT MULTIPLIERS

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ABSTRACT. The Taylor spectrum of a mixed pair of multiplication operators is determined on an ultraprime Banach algebra.

Suppose A is a complex Banach algebra, with identity 1 and invertible group A^{-1} (more generally, much of what we have to say may make sense in an additive category A); then we can define various kinds of “joint spectrum” for systems of elements in A :

Definition 1. Write

$$(1.1) \quad \sigma_A^{\text{left}}(A^n) = \{a \in A^n : 1 \notin \sum_{j=1}^n Aa_j\} \text{ and } \sigma_A^{\text{right}}(A^n) = \{a \in A^n : 1 \notin \sum_{j=1}^n a_j A\},$$

$$(1.2) \quad \pi_A^{\text{left}}(A^n) = \{a \in A^n : L_a^{-1}(0) \neq \{0\}\} \text{ and } \pi_A^{\text{right}}(A^n) = \{a \in A : R_a^{-1}(0) \neq \{0\}\}$$

and

$$(1.3) \quad \begin{aligned} & \tau_A^{\text{left}}(A^n) = \{a \in A^n : \inf_{\|x\| \geq 1} \|L_a x\| = 0\} \\ & \text{and} \\ & \tau_A^{\text{right}}(A^n) = \{a \in A^n : \inf_{\|x\| \geq 1} \|R_a x\| = 0\}; \end{aligned}$$

then for each $\omega = \sigma^{\text{left}}, \sigma^{\text{right}}, \pi^{\text{left}}, \pi^{\text{right}}, \tau^{\text{left}}, \tau^{\text{right}}$ set for arbitrary $a \in A^n$

$$(1.4) \quad \omega_A(a) = \{\lambda \in \mathbb{C}^n : a - \lambda \in \omega_A(A^n)\}.$$

This slightly eccentric notation reproduces the definitions ([4], Definitions 1.1 - 1.3) of the left and right spectrum, point and approximate spectrum of an n -tuple of Banach algebra elements. We shall also write

$$(1.5) \quad \omega_A(a) = \omega_A^{\text{left}}(a) \cup \omega_A^{\text{right}}(a)$$

for each $\omega = \sigma, \tau, \pi$. We have, for each $a \in A^n$ and $x \in A$,

$$(1.6) \quad L_a(x) = ax \in A^n \text{ and } R_a(x) = xa \in A^n,$$

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giving the product A^n any one of the obvious norms. Thus

$$(1.7) \quad L_a^{-1}(0) = \bigcap_j \{x \in A : a_j x = 0\} \text{ and } R_a^{-1}(0) = \bigcap_j \{x \in A : x a_j = 0\} :$$

in an *abus de notation* we shall find it convenient to write also

$$(1.8) \quad L_a(A^n) = \sum_j a_j A \text{ and } R_a(A^n) = \sum_j A a_j.$$

We recall how these spectra behave under homomorphisms and polynomial mappings:

Lemma 2. *If $a \in A^n$ and if $p : A^n \rightarrow A^m$ is a polynomial system then there is inclusion*

$$(2.1) \quad p\omega_A(a) \subseteq \omega_{Ap}(a),$$

with equality for commuting systems of elements when ω is either spectrum or approximate point spectrum. If $T \in HBL(A, B)$ is a bounded homomorphism between Banach algebras (preserving identity) then there is inclusion

$$(2.2) \quad \omega_B(Ta) \subseteq \omega_A(a) \text{ for } \omega = \sigma^{\text{left}}, \sigma^{\text{right}}.$$

If in addition T is one-one there is inclusion

$$(2.3) \quad \omega_A(a) \subseteq \omega_B(Ta) \text{ for } \omega = \pi^{\text{left}}, \pi^{\text{right}},$$

and if in addition T is bounded below inclusion

$$(2.4) \quad \omega_A(a) \subseteq \omega_B(Ta) \text{ for } \omega = \tau^{\text{left}}, \tau^{\text{right}}.$$

Proof. This is well known ([3], Theorem 3.2; [7], Theorems 11.2.2, 11.3.4, 11.1.4). □

In the particular case $B = BL(A, A)$ of the algebra of bounded linear operators on A and either the homomorphism $T : a \mapsto L_a$ or the antihomomorphism $T : a \mapsto R_a$ of (1.5) we can say more:

Lemma 3. *If $a \in A^n$ is arbitrary there are equalities*

$$(3.1) \quad \pi_A^{\text{left}}(a) = \pi_B^{\text{left}}(L_a) \text{ and } \pi_A^{\text{right}}(a) = \pi_B^{\text{left}}(R_a),$$

$$(3.2) \quad \tau_A^{\text{left}}(a) = \tau_B^{\text{left}}(L_a) \text{ and } \tau_A^{\text{right}}(a) = \tau_B^{\text{left}}(R_a)$$

and

$$(3.3) \quad \sigma_A^{\text{left}}(a) = \pi_B^{\text{right}}(R_a) = \tau_B^{\text{right}}(R_a) \text{ and } \sigma_A^{\text{right}}(a) = \pi_B^{\text{right}}(L_a) = \tau_B^{\text{right}}(L_a).$$

Proof. This is again familiar ([4], Theorem 2.4; [8], Theorem 5.6.3). □

From Lemma 2 and Lemma 3 we can make some observations about joint spectra for a mixed system $(L_a, R_b) \in B^n \times B^m$ induced by $(a, b) \in A^n \times A^m$:

Theorem 4. *If $a \in A^n$ and $b \in A^m$ is arbitrary then, with $B = BL(A, A)$, there are inclusions*

$$(4.1) \quad \sigma_B^{\text{left}}(L_a, R_b) \subseteq \sigma_A^{\text{left}}(a) \times \sigma_B^{\text{right}}(b) \text{ and } \sigma_B^{\text{right}}(L_a, R_b) \subseteq \sigma_A^{\text{right}}(a) \times \sigma_B^{\text{left}}(b),$$

$$(4.2) \quad \tau_B^{\text{left}}(L_a, R_b) \subseteq \tau_A^{\text{left}}(a) \times \tau_B^{\text{right}}(b) \text{ and } \tau_B^{\text{right}}(L_a, R_b) \subseteq \tau_A^{\text{right}}(a) \times \tau_B^{\text{left}}(b)$$

and

$$(4.3) \quad \pi_B^{\text{left}}(L_a, R_b) \subseteq \pi_A^{\text{left}}(a) \times \pi_B^{\text{right}}(b) \text{ and } \pi_B^{\text{right}}(L_a, R_b) \subseteq \pi_A^{\text{right}}(a) \times \pi_B^{\text{left}}(b).$$

Proof. This is (2.1) from Lemma 2 together with Lemma 3. □

When in particular A itself is an algebra of operators then ([5], Theorem 3.2) we can say more; what we can say extends to algebras A which are “ultraprime”:

Definition 5. The Banach algebra A is said to be prime if there is implication, for arbitrary a, b in A ,

$$(5.1) \quad L_a R_b = 0 \Rightarrow 0 \in \{a, b\},$$

and we shall say that A is ultraprime if there is equality, for arbitrary a, b in A ,

$$(5.2) \quad \|L_a R_b\| = \|a\| \|b\|.$$

We shall call A spectrally prime if there is equality, for arbitrary a, b in A ,

$$(5.3) \quad \sigma_B(L_a R_b) = \sigma_A(a) \sigma_A(b).$$

It is clear at once that ultraprime algebras are prime; Mathieu ([10], Proposition 2.3) has shown that for C^* algebras all three conditions are equivalent.

To see how Lemma 3 improves for ultraprime algebras we introduce two auxiliary spectrum-like objects, writing (cf. (3.2.4) and (3.2.5) of [5]) for arbitrary $a \in A^n$ and $b \in A^m$

$$(5.4) \quad \varpi^{\text{left, right}}(A^n \times A^m) = \{(a, b) \in A^n \times A^m : \inf_{\|xyz\| \geq 1} \|ax\| \|z\| \|y\| + \|x\| \|z\| \|yb\| = 0\}$$

and

$$(5.5) \quad \varpi^{\text{right, left}}(A^n \times A^m) = \{(a, b) \in A^n \times A^m : \inf_{\|xyz\| \geq 1} \|xa\| \|z\| \|y\| + \|x\| \|z\| \|by\| = 0\},$$

and then setting

$$(5.6) \quad \omega_A(a, b) = \{(\lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^m : (a - \lambda, b - \mu) \in \varpi(A^n \times A^m)\}$$

in both cases.

Theorem 6. *If $B = BL(A, A)$ there are inclusions, for arbitrary $a \in A^n$ and $b \in A^m$,*

$$(6.1) \quad \varpi_A^{\text{left, right}}(a, b) \subseteq \tau_B^{\text{left}}(L_a, R_b)$$

and

$$(6.2) \quad \varpi_A^{\text{right, left}}(a, b) \subseteq \tau_B^{\text{right}}(L_a, R_b).$$

If in particular the algebra A is ultraprime then there are also inclusions

$$(6.3) \quad \tau_A^{\text{left}}(a) \times \tau_A^{\text{right}}(b) \subseteq \varpi_A^{\text{left, right}}(a, b)$$

and

$$(6.4) \quad \tau_A^{\text{right}}(a) \times \tau_A^{\text{left}}(b) \subseteq \varpi_A^{\text{right, left}}(a, b).$$

Proof. This is the argument of Theorem 3.2 of [5] (cf. [8], Theorem 5.6.4): if $(0, 0)$ is not in the right-hand side of (6.1) there are $k > 0$ and $h > 0$ for which, for arbitrary $w \in A$,

$$(6.5) \quad \|w\| \leq k \|aw\| + h \|wb\| :$$

now substitute xyz for w . If $(0, 0)$ is not in the right-hand side of (6.2) then there are $k > 0$ and $h > 0$ for which, for arbitrary $z \in A$, there are $a' \in A^n$ and $b' \in A^m$ for which

$$(6.6) \quad \sum_j a_j a'_j + \sum_i b'_i b_i = z \text{ with } \|a'\| \leq k\|z\| \text{ and } \|b'\| \leq h\|z\| :$$

now multiply on either side by x and y . Notice an application of the open mapping theorem here, a point which was missed in the argument of Theorem 3.2.4 of [5].

Conversely if $(0, 0)$ is in the left-hand side of (6.3) then there are sequences (x_r) and (y_r) in A for which $\|x_r\| = \|y_r\| = 1$ with $\|L_a x_r\| + \|R_b y_r\| \rightarrow 0$, and then by ultraprimehood (z_r) in A for which $\|z_r\| \leq 2$ with $\|x_r z_r y_r\| = 1$. This puts $(0, 0)$ in the right-hand side of (6.3); the proof of (6.4) is similar \square

If we only assume that A is prime there is an analogue of Theorem 6 in which τ is replaced by π : we leave the reader to write down the analogues of (5.5) and (5.6) that are appropriate. From Theorem 6 we can deduce a spectral mapping theorem for polynomials in L_a and R_b :

Theorem 7. *If A is ultraprime and if $a \in A^n$ and $b \in A^m$ are arbitrary then for arbitrary systems of polynomials $p : A^n \times A^m \rightarrow A^q$ there are equalities*

$$(7.1) \quad \tau_B^{\text{left}} p(L_a, R_b) = p(\tau_A^{\text{left}}(a), \tau_A^{\text{right}}(b)) \text{ and } \tau_B^{\text{right}} p(L_a, R_b) = p(\tau_A^{\text{right}}(a), \tau_A^{\text{left}}(b)).$$

In particular for single elements $a = a_1 \in A$ and $b = b_1 \in A$ and for one polynomial $p = p_1$ in two variables there is equality

$$(7.2) \quad \sigma_B p(L_a, R_b) = p(\sigma_A(a), \sigma_A(b)).$$

It follows that there is implication

$$(7.3) \quad A \text{ ultraprime} \Rightarrow A \text{ spectrally prime.}$$

Proof. The first part is clear from Theorem 6; for the second recall ([5], Theorem 1.3; [8], Theorem 11.6.7) that for compact sets K and H in \mathbf{C} and one polynomial in two variables there is equality

$$(7.4) \quad p(\partial K, \partial H) = p(\partial(K \times H)) = p(K \times H).$$

For the implication (7.3) apply (7.2) with the polynomial $p(z, w) = zw$. \square

When the algebra A is prime, or ultraprime, then it is of interest to try and determine the ‘‘Taylor spectrum’’ and the ‘‘Taylor split spectrum’’ of systems (L_a, R_b) on the Banach space A . These are defined through the medium of the *Koszul complex*:

Definition 8. If $T \in BL(X, X)^n$ is a tuple of bounded linear operators on a Banach space X the the Koszul complex of T may be realised as the operator

$$(8.1) \quad \Lambda(T) : \sum_k x_k dz_k \mapsto \sum_{j=1}^n \sum_k T_j x_k dz_j \wedge dz_k$$

acting on the tensor product $X \otimes \Lambda(dz)$ of the space X with the exterior algebra on n generators dz_j ; for a true complex

$$(8.2) \quad \Lambda(T)^2 = \mathbf{0}$$

we need the operators T_j to commute with one another. Whether or not this is the case we shall call the system T Taylor non-singular iff ([11]; [6]; [8], Definition 11.9.3)

$$(8.3) \quad \Lambda(T)^{-1}(0) \subseteq \Lambda(T)\Lambda(X, dz),$$

and Taylor invertible ([6]; [8], Definition 11.9.3) iff there are \mathbf{U} and \mathbf{V} on $\Lambda(X, dz)$ for which

$$(8.4) \quad \mathbf{V}\Lambda(T) + \Lambda(T)\mathbf{U} = \mathbf{I}.$$

We shall write, with $B = BL(X, X)$,

$$(8.5) \quad \text{Sp}_X(T) = \tau_B^{\text{Taylor}}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda I \text{ is not Taylor nonsingular}\}$$

and

$$(8.6) \quad \text{Sp}_X^{\text{split}}(T) = \sigma_B^{\text{Taylor}}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda I \text{ is not Taylor invertible}\}.$$

Note the inclusion ([11]; [8], Theorem 11.9.9)

$$(8.7) \quad \tau_B^{\text{Taylor}}(T) \subseteq \sigma_B^{\text{Taylor}}(T) \subseteq \prod_j \sigma_B(T_j).$$

Exactness of the Koszul complex of a system of operators induces a lot of exactness between the operators themselves (*cf.* [9], Theorem 4):

Lemma 9. *If $T \in B^n$ is Taylor non-singular and $\{k\} \subseteq \{1, 2, \dots, n\}$ then there is inclusion*

$$(9.1) \quad \bigcap_{j \notin \{k\}} T_j^{-1}(0) \subseteq \sum_{j \in \{k\}} T_j(X).$$

Proof. We are writing $\{k\} = \{k_1, k_2, \dots, k_p\}$ if $k = k_1 k_2 \dots k_p$ is a multiindex. If $x_k \in X$ then by (8.3)

$$\Lambda(T)(x_k dz_k) = 0 \Rightarrow x_k dz_k \in \Lambda(T)\Lambda(X, dz)$$

which means that

$$\sum_{j \notin \{k\}} T_j x_k dz_{jk} = 0 \Rightarrow x_k dz_{jk} = \sum_{j \ell = \pm k} T_j x_\ell dx_k.$$

□

Two special cases are familiar: if $\{k\} = \emptyset$ this says that the intersection of the null spaces of the T_j is $\{0\}$, while if $\{k\} = \{1, 2, \dots, n\}$ this says that the sum of the ranges of the T_j is X .

For pairs of operators $T \in B^2$ (commuting or not) we can write

$$(9.2) \quad \tau_B^{\text{Taylor}}(T) = \tau_B^{\text{left}}(T) \cup \tau_B^{\text{middle}}(T) \cup \tau_B^{\text{right}}(T),$$

where

$$(9.3) \quad (0, 0) \notin \tau_B^{\text{middle}}(T) \Leftrightarrow (\Lambda_1(T), \Lambda_0(T)) \text{ not exact.}$$

For a pair of bounded linear operators $U : X \rightarrow Y$ and $V : Y \rightarrow Z$ between Banach spaces (whether or not $VU = 0$) “exactness” means

$$(9.4) \quad V^{-1}(0) \subseteq U(X) \subseteq Y \text{ with } V(Y) = \text{cl } V(Y) \subseteq Z :$$

equivalently ([7], Theorem 2.3; [8], Theorem 10.5.4), there are $k > 0$ and $h > 0$ for which for arbitrary $y \in Y$ there is $x \in X$ satisfying

$$(9.5) \quad \|y - Ux\| \leq k\|Vy\| \text{ and } \|x\| \leq h\|y\|.$$

Towards a determination of the Taylor spectrum of a pair (L_a, R_b) in an ultraprime Banach algebra A , we begin with no assumption:

Theorem 10. *If x, y are in A and $a \in A^n, b \in A^m$ there are implications*

$$(10.1) \quad x \in L_a^{-1}(0), y \in R_b^{-1}(0) \Rightarrow (L_x R_y)(A) \subseteq L_a^{-1}(0) \cap R_b^{-1}(0);$$

$$(10.2) \quad x \in R_a^{-1}(0), y \in L_b^{-1}(0) \Rightarrow L_a(A^n) + R_b(A^m) \subseteq (L_x R_y)^{-1}(0);$$

$$(10.3) \quad x \in L_a^{-1}(0), y \in L_b^{-1}(0) \Rightarrow (L_x R_y)(A) \subseteq L_a^{-1}(0) L_b^{-1}(0);$$

$$(10.4) \quad x \in R_a^{-1}(0), y \in R_b^{-1}(0) \Rightarrow (L_x R_y)(A) \subseteq R_a^{-1}(0) R_b^{-1}(0).$$

Proof. Clear (but remember the notation (1.7) and (1.8)). \square

When the algebra A is prime then all the point spectrum of a and b is in the spectrum of (L_a, R_b) :

Theorem 11. *If A is prime then for arbitrary $a \in A^n$ and $b \in A^m$ there is inclusion*

$$(11.1) \quad \pi_A(a) \times \pi_A(b) \subseteq \tau_B^{\text{Taylor}}(L_a, R_b).$$

Proof. To see this break the product $\pi_A(a) \times \pi_A(b)$ into the union of four products of left and right spectra and then combine each of the four implications of Theorem 10 with the assumption that A is prime. Thus if $(0, 0) \in \pi_A^{\text{left}}(a) \times \pi_A^{\text{right}}(b)$ then there are $x, y, z \in A$ with $ax = yb = 0 \neq xzy$, so that by (10.1) $L_a^{-1}(0) \cap R_b^{-1}(0) \neq \{0\}$. If instead $(0, 0) \in \pi_A^{\text{right}}(a) \times \pi_A^{\text{left}}(b)$ then there are $x, y, z \in A$ with $xa = by = 0 \neq xzy$, so that by (10.2) $L_a(A^n) + R_b(A^m) \neq A$. If instead $(0, 0) \in \pi_A^{\text{left}}(a) \times \pi_A^{\text{left}}(b)$ then there are $x, y, z \in A$ with $ax = by = 0 \neq xzy$, so that by (10.3) $L_a^{-1}(0) L_b^{-1}(0) \neq \{0\}$. Since of course $R_b(A) L_b^{-1}(0) = \{0\}$ this means $L_a^{-1}(0) \not\subseteq R_b(A^m)$; now recall (9.1). The argument for the fourth and final case is exactly the same. \square

We have used here the argument of Curto and Hernández ([3], proof of Theorem 2): notice the inclusions

$$(11.2) \quad \sigma_A^{\text{left}}(a) \setminus \sigma_A^{\text{right}}(a) \subseteq \pi_A^{\text{left}}(a) \text{ and } \sigma_A^{\text{right}}(a) \setminus \sigma_A^{\text{left}}(a) \subseteq \pi_A^{\text{right}}(a).$$

It would be nice if at this point we could pass from the Banach algebra A to its “enlargement”

$$(11.3) \quad \mathbf{Q}(A) = \ell_\infty(A)/c_0(A),$$

so that the approximate point spectra of a and b would coincide with the point spectrum of their images in the enlargement, while the Taylor spectrum of (L_a, R_b) would remain unchanged: we would want the enlargement of an ultraprime algebra to be prime. Unfortunately with this simple-minded concept of enlargement that cannot happen: if $A \neq \mathbf{0}$ then the sequences

$$(11.4) \quad a = (1, 0, 1, 0, \dots), \quad b = (0, 1, 0, 1, \dots)$$

satisfy

$$(11.5) \quad a \notin c_0(A) ; b \notin c_0(A) ; a\ell_\infty(A)b = \{0\} \subseteq c_0(A).$$

Thus we need to get our hands dirty, and look at exactness (9.5). When the pair $(V, U) = \left(\begin{pmatrix} -S & T \end{pmatrix}, \begin{pmatrix} T \\ S \end{pmatrix} \right)$ forms the Koszul complex of a pair of operators (S, T) on a Banach space X then as in Lemma 9 a necessary condition for exactness is that each of the pairs (S, T) and (T, S) is also exact, this time in the sense of (9.5):

Theorem 12. *If A is ultraprime then for arbitrary $a \in A$ and $b \in A$ there is inclusion*

$$(12.1) \quad \tau_A(a) \times \tau_A(b) \subseteq \tau_B^{\text{Taylor}}(L_a, R_b) \subseteq \sigma_A(a) \times \sigma_A(b).$$

Proof. The second inclusion is (8.7). Towards the first we already know by Theorem 6 that two of the products are included:

$$(12.2) \quad \tau_A^{\text{left}}(a) \times \tau_A^{\text{right}}(b) \subseteq \tau_B^{\text{left}}(L_a, R_b)$$

and

$$(12.3) \quad \tau_A^{\text{right}}(a) \times \tau_A^{\text{left}}(b) \subseteq \tau_B^{\text{right}}(L_a, R_b).$$

We claim

$$(12.4) \quad \tau_A^{\text{left}}(a) \times \tau_A^{\text{left}}(b) \subseteq \tau_B^{\text{middle}}(L_a, R_b)$$

and

$$(12.5) \quad \tau_A^{\text{right}}(a) \times \tau_A^{\text{right}}(b) \subseteq \tau_B^{\text{middle}}(L_a, R_b).$$

Thus suppose for a contradiction that $(0, 0)$ is in the left-hand side of (12.4) but not in the right: thus there are (x_r) and (y_r) in A for which

$$(12.6) \quad \|x_r\| = \|y_r\| = 1 \text{ with } \|ax_r\| + \|by_r\| \rightarrow 0.$$

Now if $(0, 0)$ is not in the right-hand side of (12.4) then both (R_b, L_a) and (L_a, R_b) are exact in the sense of (9.5). Taking the first alternative there is, for arbitrary $z \in A, w_r \in A$ for which

$$(12.7) \quad \|x_r z - w_r b\| \leq k \|ax_r z\| \text{ with } \|w_r\| \leq h \|x_r z\| :$$

it follows

$$\begin{aligned} \|x_r z y_r\| &\leq \|x_r z - w_r b\| \|y_r\| + \|w_r\| \|by_r\| \leq \\ &k \|ax_r\| \|z\| \|y_r\| + h \|x_r\| \|z\| \|by_r\| \rightarrow 0. \end{aligned}$$

By uniform boundedness ([8], Theorem 4.9.1) it follows that

$$\sup_{\|z\| \leq 1} \|x_r z y_r\| \rightarrow 0,$$

and then by ultraprimiteness $\|y_r\| \|x_r\| \rightarrow 0$, a contradiction. If instead $(0, 0)$ is in the left-hand side of (12.5) then apply (9.5) with L_a and R_b reversed. \square

If either $A = BL(X, X)$ or A is a C^* -algebra, so that there is equality

$$(12.8) \quad \sigma_A(a) = \tau_A(a) \text{ for each } a \in A,$$

then of course [3] there is equality throughout (12.1). An extension of the first inclusion of (12.1) to $a \in A^n$ and $b \in A^m$ would follow from the analogue of Lemma 9 with the exactness condition (9.5).

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