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SUBNORMAL SUBGROUPS OF GROUP RING UNITS

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Abstract. Let G be an arbitrary group. If $a \in \mathbb{Z}G$ satisfies $a^2 = 0$, $a \neq 0$, then the units $1+a$, $1+a^*$ generate a nonabelian free subgroup of units. As an application we show that if G is contained in an almost subnormal subgroup V of units in $\mathbb{Z}G$ then either V contains a nonabelian free subgroup or all finite subgroups of G are normal. This was known before to be true for finite groups G only.

0. Introduction

Let $\mathbb{Z}G$ denote the integral group ring of a group G. Let $\mathcal{U}_1\mathbb{Z}G$ be the group of invertible elements in this ring which are of augmentation one: if $u = \sum u_g g$ then $\sum u_g = 1$.

Let G be an arbitrary group. Clearly $G \leq \mathcal{U}_1\mathbb{Z}G$. If not all finite subgroups of G are normal in G then we are able to construct a nontrivial unit $u \in \mathcal{U}_1\mathbb{Z}G$, $u \notin G$, in the following way. Choose $x, y \in G$ so that $o(x) = n < \infty$ and $y\langle x\rangle y^{-1} \neq \langle x\rangle$. For $a = (1-x)y(1+x+\cdots+x^{n-1}) \in \mathbb{Z}G$ we have $a \neq 0$ and $a^2 = 0$. Hence $u = 1+a$ is a unit and $u^{-1} = 1-a$. Units obtained in this way are called *bicyclic units*.

It is well known that $\mathcal{U}_1\mathbb{Z}G$ quite often contains a nonabelian free subgroup. In fact, the following theorem was proved by Sehgal [13, p. 200] and also by Hartley–Pickel [4] (see also [14, p. 19]).

0.1. Theorem. *Let G be a solvable group which has a non normal finite subgroup. Then $\mathcal{U}_1\mathbb{Z}G$ contains a nonabelian free subgroup.*

Also, all nonabelian finite groups G, except Hamiltonian 2-groups, have a free subgroup inside $\mathcal{U}_1\mathbb{Z}G$.

Free subgroups in $\mathcal{U}_1\mathbb{Z}G$ were also studied by Jespers who characterized in [5] all finite groups G which have a free complement in $\mathcal{U}_1\mathbb{Z}G$. Jespers, Leal and del Río classified in [6] all finite nilpotent groups G such that $\mathcal{U}_1\mathbb{Z}G$ has a subgroup of finite index which is a direct product of noncyclic free groups. Recently Leal and del Río [7] removed the nilpotent condition and classified all finite groups with the same property.

These results were obtained by a careful study of rational representations of finite groups. Except a few examples, these were really existence results.

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Recently, we have found an elementary and explicit construction of a free subgroup inside $\mathcal{U}_1\mathbb{Z}G$, also for infinite groups. Namely, we proved in [10] that if $u \in \mathbb{Z}G$ is a bicyclic unit then $\langle u, u^*\rangle$ is a nonabelian free subgroup of $\mathcal{U}_1\mathbb{Z}G$, where $(\sum a_g g)^* = \sum a_g g^{-1}$.

Also assuming the existence of bicyclic units, Goncalves and Passman gave in [3] a construction of a free subgroup inside $\mathcal{U}_1 KG$ when K is a field, char$K = p > 0$, tr.deg.$K > 0$ and G has no p-torsion.

In the present paper we extend the main result of [10] in the following way.

0.2. Theorem. *Let $a \in \mathbb{Z}G$ be an arbitrary element satisfying $a \neq 0$, $a^2 = 0$. Set $u = 1 + a$. Then the subgroup $\langle u, u^*\rangle < \mathcal{U}_1\mathbb{Z}G$ is free and nonabelian.*

We apply this theorem to generalize the main result of [2]. Let us say that a subgroup $V < \mathcal{U}_1\mathbb{Z}G$ is *almost subnormal* if V is subnormal in a finite index subgroup of $\mathcal{U}_1\mathbb{Z}G$.

0.3. Theorem (Goncalves-Ritter-Sehgal, [2]). *Let G be a finite group. If $V < \mathcal{U}_1\mathbb{Z}G$ is an almost subnormal subgroup containing G then either*

(i) *V has a nonabelian free subgroup, or*

(ii) *G is abelian or a Hamiltonian 2–group.*

The original proof depended on the classification of finite groups all of whose proper subgroups and factor groups are Hamiltonian 2–groups or abelian, followed by a careful investigation of their rational representations.

Notice that all subgroups of groups mentioned in (ii) are normal. Keeping this in mind, we offer the following generalization:

0.4. Theorem. *Let G be an arbitrary group. If $V < \mathcal{U}_1\mathbb{Z}G$ is an almost subnormal subgroup containing G then either*

(i) *V has a nonabelian free subgroup, or*

(ii) *all finite subgroups of G are normal.*

1. Free subgroups in $\mathcal{U}_1\mathbb{Z}G$

We construct here some free subgroups inside $\mathcal{U}_1\mathbb{Z}G$. In fact, it is equally easy to do this in a more general context.

Let A be a $\mathbb{C}$-algebra with an involution denoted by $*$. Following [12, p. 302] we say that a Hermitian inner product $(\text{-},\text{-})\colon A \times A \longrightarrow \mathbb{C}$ is compatible with $*$ if $(a^*b, c) = (b, ac)$ holds for all $a, b, c \in A$.

1.1. Examples. (a) Let $A = \mathbb{C}G$, the group algebra of an arbitrary group G with the standard involution $(\sum a_g g)^* = \sum \overline{a}_g g^{-1}$. The Hermitian inner product compatible with $*$ is defined by $(a, b) = \text{tr}(a^*b)$ where tr is the group ring trace: $\text{tr}(\sum a_g g) = a_1$ (compare [11, p.33]).

(b) Let $A = M_n(\mathbb{C})$, the matrix ring with the involution $a^* = \overline{a}^T$. The Hermitian inner product compatible with $*$ is given by $(a, b) = \text{Tr}(a^*b)$ where Tr is the usual matrix trace.

(c) Let A be any H^*-algebra. It is a Banach $*$-algebra whose norm is a Hilbert space norm such that $(a, bc^*) = (ac, b) = (c, a^*b)$ for all $a, b, c \in A$ (see [1]).

We can equip any such algebra with a norm $\|.\|\colon A \longrightarrow \mathbb{R}$ defined by $\|a\| = (a, a)^{1/2}$.

The main result of this section is

1.2. Theorem. *Let A be a $\mathbb{C}$-algebra with an involution $*$ and with a compatible Hermitian inner product $(\text{-},\text{-})$. If an element $a \in A$ satisfies $a^2 = 0$ and $\|a^*a\| \geq 4 \cdot \|1\|$ then $\langle 1+a, 1+a^*\rangle$ is a free nonabelian subgroup of units in A.*

For the proof we consider the sequence $T = \{1, c, c^2, c^3, \dots\}$ of elements in A, where $c = a^*a \in A$. In what follows we will assume that $\|c\| \geq 4 \cdot \|1\|$. Our argument splits into two cases depending on the linear dependence of T.

1.3. Lemma. *If the sequence T is linearly independent then*

(i) *the submonoid S of $(A, \cdot)$ generated by $\{a, a^*\}$ is isomorphic to $\langle s, t | s^2 = 0 = t^2\rangle$,*

(ii) *the linear span of S in A is isomorphic to the (reduced) semigroup algebra $\mathbb{C}S$.*

Proof. Let Σ be the monoid given by the presentation $\langle s, t | s^2 = 0, t^2 = 0\rangle$. Consider the monoid map $\phi \colon \Sigma \longrightarrow S$ given by $\phi(s) = a^*$, $\phi(t) = a$.

To prove (i) it is enough to show that ϕ is injective. To this end we decompose Σ into five pairwise disjoint pieces: $\Sigma = \{0\} \cup \mathcal{T} \cup t\mathcal{T} \cup \mathcal{T}s \cup t\mathcal{T}s$, where $\mathcal{T} = \{\, (st)^k \mid k \geq 0\}$. Notice that ϕ is injective on $\mathcal{T} \subset \Sigma$ as, by assumption, all powers of c are distinct.

Suppose that $\phi(x) = \phi(y)$ for two distinct elements $x, y \in \Sigma \setminus \mathcal{T}$. If x, y belong to the same part of Σ, we can multiply them both on the left by s and/or on the right by t to obtain two distinct elements of $\mathcal{T}$ mapping to the same element of S. This is a contradiction with the last paragraph. On the other hand, if x, y belong to different parts of Σ then a simultaneous left and/or right multiplication gives us a new pair $x' = 0$, $y' \in \mathcal{T}$. But then $0 = \phi(x') = \phi(y') \in T$ – a contradiction, as T is linearly independent. This proves (i).

To prove (ii) it is enough to show that $S \setminus \{0\}$ is linearly independent in A. Consider any linear combination of elements from $S \setminus \{0\}$. We can write it in the form $\alpha + \beta + \gamma + \delta = 0$ where the four summands are combinations of elements from T, aT, Ta^* and aTa^* respectively. Note that these sets cover $S \setminus \{0\}$ and are pairwise disjoint by part (i).

If $\alpha \not\equiv 0$ then we multiply our combination on both sides by a^*a. We obtain a non-trivial combination $(a^*a)\alpha(a^*a) = 0$ of elements from T, a contradiction. Hence $\alpha \equiv 0$ and so $\beta + \gamma + \delta = 0$.

We can continue in this way to show that also $\beta \equiv \gamma \equiv \delta \equiv 0$. For details look in the proof of Lemma 6 in [10]. □

1.4. Lemma. *If the sequence T is linearly dependent then the linear map $f : A \longrightarrow A$ given by $f(x) = cx$ has an eigenvalue λ with $|\lambda| \geq 4$.*

Proof. Choose a maximal positive integer d so that the set $T_d = \{1, c, \dots, c^{d-1}\}$ is linearly independent. Let W be the $\mathbb{C}$-linear span of T_d in A. Then $c^d \in W$ and hence the finite dimensional subspace W is invariant under f.

Notice that $c^* = c$. Then for $v, w \in W$ we have $(v, f(w)) = (v, cw) = (c^*v, w) = (cv, w) = (f(v), w)$, i.e. $f \colon W \longrightarrow W$ is a Hermitian operator with respect to the Hermitian inner product $(\text{-},\text{-})_{|W}$. From the Spectral Theorem (see [9, Ch.XIV, §12]) it follows that W has an orthonormal basis $v_1, \dots, v_d$ consisting of eigenvectors of f.

Let $f(v_i) = \lambda_i v_i$ for $i = 1, \dots, d$. We prove that we must have $|\lambda_i| \geq 4$ for at least one i. To this end write $1 \in W \subset A$ as a combination of the eigenvectors: $1 = \sum \alpha_i v_i$, $\alpha_i \in \mathbb{C}$. Then $c = c \cdot 1 = \sum \alpha_i c v_i = \sum \alpha_i f(v_i) = \sum \alpha_i \lambda_i v_i$. Hence $\|c\|^2 = (c, c) = \sum |\alpha_i \lambda_i|^2$, by the Pythagorean Theorem.

Suppose that $|\lambda_i| < 4$ holds for $i = 1, \dots, d$. Then $\|c\|^2 < 16 \cdot \sum |\alpha_i|^2 = 16 \cdot \|1\|^2$ and so $\|c\| < 4 \cdot \|1\|$ – a contradiction with the assumption that $\|c\| \geq 4 \cdot \|1\|$. □

Proof of Theorem 1.2. We look at the sequence T. If T is linearly independent then, by Lemma 1.3, the subalgebra $R \subset A$ generated by $\{a^*, a\}$ is isomorphic to the semigroup algebra $\mathbb{C}\Sigma$. Consider the monoid map $\rho\colon \Sigma \longrightarrow M_2(\mathbb{C})$ given by $\rho(s) = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$, $\rho(t) = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$. We extend ρ by linearity to an algebra homomorphism $\overline{\rho}\colon R \approx \mathbb{C}\Sigma \longrightarrow M_2(\mathbb{C})$. Then $\overline{\rho}(1 + a^*) = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $\overline{\rho}(1 + a) = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and hence $1 + a^*$, $1 + a$ generate together a nonabelian free subgroup, by a well known theorem of Sanov:

1.5. Theorem (Sanov, see [8, p. 92]). *For any complex number z satisfying $|z| \geq 2$ the matrices $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$ generate a nonabelian free subgroup of $GL_2(\mathbb{C})$.*

We are thus left with the case when T is linearly dependent. By Lemma 1.4 there exists an element $v \in A$, $v \neq 0$, such that $a^* a \cdot v = \lambda v$ and $|\lambda| \geq 4$. Choose $\mu \in \mathbb{C}$ so that $\mu^2 = \lambda$ and define $w = (1/\mu) a \cdot v \in A$.

The subspace $V = \operatorname{span}_{\mathbb{C}}(v, w) \subset A$ is invariant under left multiplication by a^* and by a. In fact,

$$a^* \cdot v = (1/\lambda)(a^*)^2 a v = 0, \quad a \cdot v = \mu w,$$
$$a^* \cdot w = (1/\mu) a^* a v = \mu v, \quad a \cdot w = (1/\mu) a^2 v = 0.$$

From the above equalities it follows that the vectors v, w are linearly independent and hence they form a basis of V.

Obviously, left multiplications by $1 + a^*$ and $1 + a$ also preserve the subspace V. With respect to the basis $\{v, w\}$ they are represented by the matrices

$$1 + a^* \leftrightarrow \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \qquad 1 + a \leftrightarrow \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}.$$

However, we have $|\lambda| \geq 4$ and hence $|\mu| \geq 2$. Therefore the above pair of matrices generates a free group, by Theorem 1.5. Hence the units $1 + a^*, 1 + a \in A$ do the same. □

Now we specialize to group algebras.

1.6. Theorem. *If an element $a \in \mathbb{C}G$ satisfies $a^2 = 0$ and $\|a\| \geq 2$ then $\langle 1 + a, 1 + a^* \rangle$ is a free nonabelian subgroup of units in $\mathbb{C}G$.*

Proof. Let $c = a^* a$. Then we have $\|c\|^2 = (c, c) = \operatorname{tr}(c^* c) = \sum |c_g|^2 \geq |c_1|^2$. It follows that $\|c\| \geq |c_1| = |\operatorname{tr}(c)| = |\operatorname{tr}(a^* a)| = |(a, a)| = \|a\|^2 \geq 4$. Hence $\|a^* a\| \geq 4 = 4 \cdot \|1\|$ and we may apply Theorem 1.2. □

1.7. Theorem. *If an element $a \in \mathbb{Z}G$ satisfies $a^2 = 0$ and $a \neq 0$ then $\langle 1 + a, 1 + a^* \rangle$ is a free nonabelian subgroup of units in $\mathbb{Z}G$.*

Proof. By the previous theorem it is enough to show that $\|a\| \geq 2$. Suppose then that $\|a\| < 2$, i.e. $4 > \|a\|^2 = \sum a_g^2$. Hence all a_g belong to $\{-1, 0, 1\}$ and a has at most three elements in its support. From $a^2 = 0$ it follows that a has augmentation zero and hence $a = g - h$ for some $g, h \in G$. Then from $0 = a^2 = g^2 + h^2 - gh - hg$ it easily follows that $g = h$ and $a = 0$ – a contradiction. □

1.8. Corollary. *If $\mathbb{Z}G$ contains a nonzero nilpotent element then the unit group of $\mathbb{Z}G$ contains a nonabelian free subgroup.*

1.9. Corollary. *Suppose that the group ring $\mathbb{Z}G$ is not a domain. If G has no non-identity finite normal subgroups then the group of units of $\mathbb{Z}G$ contains a nonabelian free subgroup.*

Proof. Let $x, y \in \mathbb{Z}G$ be nonzero elements such that $xy = 0$. From Connell's Theorem [11, Thm. 4.2.10] it follows that the algebra $\mathbb{Q}G$ is prime and hence $y\mathbb{Q}Gx \neq 0$. Pick an element $z \in \mathbb{Z}G$ so that $a = yzx \neq 0$. Then $a^2 = 0$ and Theorem 1.7 applies. □

2. An application

Here we prove Theorem 0.4. We start with

2.1. Lemma. *Suppose that an almost subnormal subgroup $V < \mathcal{U}_1\mathbb{Z}G$ contains G. If $\mathbb{Z}G$ has a bicyclic unit then V contains a nonabelian free subgroup.*

Proof. Write $\mathcal{U} = \mathcal{U}_1\mathbb{Z}G$ for short. We have

$$G < V = V_r \triangleleft V_{r-1} \triangleleft \cdots \triangleleft V_1 < \mathcal{U}$$

and $|\mathcal{U} : V_1| < \infty$. It follows there exists a normal subgroup $N \triangleleft \mathcal{U}$ such that $N < V_1$ and $|\mathcal{U} : N| < \infty$. Consider $N^* = \{u \in \mathcal{U} | u^* \in N\}$. Then also $|\mathcal{U} : N^*| < \infty$ and $N^* \triangleleft \mathcal{U}$. Set $\widetilde{N} = N \cap N^*$. Then we have $\widetilde{N}^* = \widetilde{N}$, $|\mathcal{U} : \widetilde{N}| < \infty$, $\widetilde{N} \triangleleft \mathcal{U}$ and $\widetilde{N} < V_1$.

Let u be a bicyclic unit and let k be the order of the image of u in $\mathcal{U}/\widetilde{N}$. Then $u^k \in \widetilde{N}$ and $(u^*)^k = (u^k)^* \in \widetilde{N}^* = \widetilde{N}$.

By construction $u = 1 - a$ where $a = (x-1)y(1 + x + \cdots + x^{n-1})$ for suitable $x, y \in G$. Consider the group ring elements $a_s = k(x-1)^s y(1 + x + \cdots + x^{n-1})$ for $s \geq 1$. Clearly $a_s^2 = 0$. Hence we have new units $u_s = 1 - a_s \in \mathcal{U}$.

Notice that $xu_s x^{-1} = 1 - xa_s$. It follows that $u_s = [x, u_{s-1}]$ for all $s \geq 2$. In fact,

$$\begin{aligned}[x, u_{s-1}] &= x \cdot (1 - a_{s-1}) \cdot x^{-1} \cdot (1 + a_{s-1}) = (1 - xa_{s-1})(1 + a_{s-1}) \\ &= 1 + a_{s-1} - xa_{s-1} = 1 - (x-1)a_{s-1} = 1 - a_s = u_s.\end{aligned}$$

Finally, we show that $u_s, u_s^* \in V_s$ for all $s \geq 1$. Clearly $u_1 = u^k$ and hence $u_1, u_1^* \in \widetilde{N} < V_1$. Assume that $u_{s-1} \in V_{s-1}$. Then $u_{s-1}x^{-1}u_{s-1}^{-1} \in V_s$ as $x^{-1} \in G < V_s \triangleleft V_{s-1}$. Then also $u_s = x \cdot u_{s-1}x^{-1}u_{s-1}^{-1} \in V_s$. The proof that $u_s^* \in V_s$ is similar.

Now, by Theorem 1.7, $\langle u_r, u_r^* \rangle = \langle 1 + a_r, 1 + a_r^* \rangle < V_r = V$ is a free subgroup. □

Proof of Theorem 0.4. If G has a finite subgroup which is not normal then $\mathcal{U}_1\mathbb{Z}G$ contains a bicyclic unit. From Lemma 2.1 it then follows that the subgroup V contains a nonabelian free subgroup. □

2.2. Corollary. *Let G be an arbitrary group. If $\mathcal{U}_1\mathbb{Z}G$ contains a subgroup V such that*

(i) *V is subnormal in a finite index subgroup of $\mathcal{U}_1\mathbb{Z}G$,*
(ii) *$G \leq V$,*
(iii) *V is nilpotent*

then all finite subgroups of G are normal.

2.3. *Remark.* Clearly, we could change *nilpotent* in the last corollary to *solvable* or any other nontrivial variety of groups. The proof remains the same.

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