

## A NOTE ON THE COHOMOLOGY OF FINITARY MODULES

U. MEIERFRANKENFELD

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ABSTRACT. Let  $G$  be a group,  $D$  a division ring and  $V$  a  $DG$ -module.  $V$  is called finitary provided that  $V/C_V(g)$  is finite dimensional for all  $g \in G$ . We investigate the first and second degree cohomology of finitary modules in terms of a local system for  $G$ .

In this note we prove the following three theorems on the cohomology of finitary modules in terms of the cohomology of a local system of subgroups:

**Theorem 1.** *Let  $G$  be a group,  $K$  a field,  $V$  a finitary  $KG$ -module and  $\mathcal{L}$  a local system of subgroups of  $G$ . Suppose that, for all  $H \in \mathcal{L}$ ,  $V$  is completely reducible as a  $KH$ -module. Then  $[V, G]$  is completely reducible as a  $KG$ -module.*

**Theorem 2.** *Let  $G$  be a group,  $D$  a division ring,  $V$  a finitary  $DG$ -module,  $\mathcal{L}$  a local system of subgroups of  $G$  and  $H$  an extension of  $V$  by  $G$  (i.e.  $H/V \cong G$ ). Suppose that the following holds for all  $L$  in  $\mathcal{L}$ :*

- (i) *The extension of  $V$  by  $L$  in  $H$  splits.*
- (ii)  *$V/C_V(L)$  is finite dimensional.*
- (iii)  *$H^1(L, V)$  is finite dimensional.*

*Then  $H$  splits over  $V$ .*

**Theorem 3.** *Let  $G$  be a group,  $D$  a division ring,  $\mathcal{L}$  a local system of subgroups of  $G$ ,  $W$  a  $DG$ -module and  $V$  a  $DG$ -submodule of  $W$  such that  $W = V + C_W(H)$  for all  $H \in \mathcal{L}$ . Then there exists a canonical  $DG$ -monomorphism from  $W/C_W(G)$  to  $[V^*, G]^*$ , where  $Y^*$  denotes the dual of a module  $Y$ .*

We remark that conditions (ii) and (iii) in Theorem 2 are automatically fulfilled if all members of  $\mathcal{L}$  are finite groups generated by elements whose order is coprime to the characteristic of  $D$ .

*Proof of Theorem 1.* Let  $H \in \mathcal{L}$ . Then  $[V, H] = [V, H, H]$  and so  $[V, G] = [V, G, G]$ . Hence we may assume that  $V = [V, G]$ . Let  $W$  be the sum of all the irreducible  $KG$ -submodules in  $V$ , where  $W = 0$  if  $G$  has no irreducible submodules in  $V$ . We need to show that  $W = V$ .

So suppose that  $V \neq W$ . Then  $[V, G] \not\subseteq W$  and we may assume that  $[V, H] \not\subseteq W$  for all  $H \in \mathcal{L}$ . Let  $H \in \mathcal{L}$  and let  $I_H$  be the set of irreducible  $KH$ -submodules  $I$  in  $[V, H]$  with  $I \not\subseteq W$ . For  $I \in I_H$  let  $m(I)$  be the supremum of all positive integers  $t$  such that  $I^t$  is isomorphic to a  $KH$ -submodule of  $V$ . Pick  $h \in H$  with

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$[I, h] \neq 0$ . Then  $m(I) \cdot \deg_I(h) \leq \deg_V(h)$ . In particular,  $m(I)$  is finite. Note that there exists a unique  $KH$ -submodule  $\hat{I}$  in  $V$  isomorphic to  $I^{m(I)}$ , namely  $\hat{I}$  is the submodule generated by all the  $H$ -submodules in  $V$  isomorphic to  $I$ . Let  $K(I) = \text{Hom}_{KH}(I, I)$  and  $d(I) = \dim_K K(I)$ . Since  $\dim_K [I, h] = \dim_{K(I)} [I, h] \cdot \dim_K K(I)$ ,  $d(I) \leq \deg_V(h)$  and so  $d(I)$  is finite. Let  $m$  be the minimum of all  $m(I)$ ,  $I \in I_H$ ,  $H \in \mathcal{L}$ , and  $d$  the minimum of all  $d(I)$ ,  $I \in I_H$ ,  $H \in \mathcal{L}$ ,  $m(I) = m$ .

Pick  $H \in \mathcal{L}$  and  $I \in I_H$  with  $m(I) = m$  and  $d(I) = d$ . Without loss  $H \leq F$  for all  $F \in \mathcal{L}$ . Let  $F \in \mathcal{L}$ . Since  $V$  is completely reducible as a  $KF$ -module, there exists  $J \in I_F$  such that  $I$  is isomorphic to a  $KH$ -submodule of  $J$ . Let  $a$  be a positive integer such that  $I^a$  is isomorphic to a  $KH$ -submodule of  $J$ . Then  $I^{a \cdot m(J)}$  is isomorphic to a  $KH$ -submodule of  $V$  and so  $a \cdot m(J) \leq m$ . By minimal choice of  $m$ ,  $m \leq m(J)$ . Thus  $a = 1$  and  $m(J) = m$ . In particular,  $\hat{I} \leq \hat{J}$  and there exists a unique  $KH$ -submodule  $U$  in  $J$  isomorphic to  $I$ . Hence  $K(J)$  acts on  $U$  and the restriction  $K(J)|_U$  of  $K(J)$  to  $U$  is contained in  $K(U)$ . Since  $\dim_K K(U) = \dim_K K(I) = d \leq \dim_K K(J)$ , we conclude that  $K(J)|_U = K(U)$ . It is now easy to see that every irreducible  $KH$ -submodule of  $\hat{I}$  lies in an irreducible  $KF$ -submodule of  $\hat{J}$ . Hence  $\langle I^F \rangle$  is an irreducible  $KF$ -module for all  $F \in \mathcal{L}$  and  $\langle I^G \rangle$  is an irreducible  $KG$ -submodule in  $V$  not contained in  $W$ . This contradiction completes the proof of Theorem 1.  $\square$

The following definition and lemma are used in the proof of Theorem 2.

**Definition 4.** (a) Let  $R$  be a ring,  $A$  a set,  $M$  an  $R$ -module and for  $a \in A$  let  $\rho_a: A \rightarrow M$  be a bijection. Then  $A$  is called an affine  $R$ -module provided that for all  $a, b, c$  in  $A$ ,  $\rho_a(b) + \rho_b(c) = \rho_a(c)$ .

(b) Let  $R$  be a ring,  $A$  and  $B$  affine  $R$ -modules and  $\pi: A \rightarrow B$ . Then  $\pi$  is called an affine  $R$ -homomorphism if for some  $a$  in  $A$  and  $b$  in  $B$ ,  $\rho_b \pi \rho_a^{-1}$  is an  $R$ -homomorphism of modules.

(c) Let  $R$  be a ring and  $A$  an affine  $R$ -module. A subset  $B$  of  $A$  is called an affine  $R$ -submodule if  $\rho_a(B)$  is an  $R$ -submodule of  $M$  for some  $a$  in  $A$ .

*Remark.* Let  $M$  be an  $R$ -module and define  $\rho_x: M \rightarrow M$ ,  $y \rightarrow y - x$ . Then  $M$  is an affine  $R$ -module. Moreover, if  $A$  is any affine  $R$ -module with  $M$  as underlying module, then for all  $a$  in  $A$ ,  $\rho_a$  is an isomorphism of affine  $R$ -modules. Finally if  $a, b$  are in  $A$  and  $C$  is a subset of  $A$ , then  $\rho_a(C) = \rho_b(C) + \rho_a(b)$  and so  $C$  is an affine submodule if and only if  $\rho_a(C)$  is the coset of an  $R$ -submodule in  $M$ .

**Lemma 5.** Let  $G$  be a group,  $R$  a ring and  $V$  an  $RG$ -module. Let  $A_G$  be the set of complements to  $V$  in  $V \rtimes G$ . Then

- (a)  $A_G$  is an affine  $R$ -module.
- (b) Let  $H \leq G$ . Then the canonical map from  $A_G$  to  $A_H$  is affine.
- (c) Let  $I_G = \{G^v | v \in V\}$ . Then  $I_G$  is an affine  $RG$  submodule of  $A_G$ ,  $I_G \cong V/C_V(G)$  and  $A_G/I_G \cong H^1(G, V)$ .

*Proof of the lemma.* Identify  $V$  and  $G$  with their images in the semidirect product  $V \rtimes G$ . So  $V \rtimes G = VG$ .

(a) Let  $M_G$  be the set of functions  $f: VG/V \rightarrow V$  with  $f(ab) = f(a)^{b^{-1}} + f(b)$  for all  $a, b$  in  $VG/V$ , i.e.  $M_G$  is the set of derivations for  $G$  on  $V$ . Note that  $M_G$  is an  $R$ -module via  $(r \cdot f)(a) = r \cdot f(a)$ . For  $K, L$  in  $A_G$  define  $\rho_K(L) \in M_G$  by  $\rho_K(L)(Va) = v$ , whenever  $a \in K$  and  $v \in V$  with  $va \in L$ . Then  $\rho_K$  is a bijection from  $A_G$  onto  $M_G$  (see for example [As, 17.1]).

Let  $K, L, N$  be in  $A_G$  and  $a$  in  $K$ . Put  $b = \rho_K(L)(Va)a$  and  $c = \rho_L(N)(Vb)b$ . Then  $Va = Vb = Vc$ ,  $b \in L$ ,  $c \in N$  and  $c = \rho_L(N)(Va)\rho_K(L)(Va)a$ . Thus  $\rho_K(L) + \rho_L(N) = \rho_K(N)$ . (Here we write the binary operation on  $V$  multiplicatively whenever  $V$  is regarded as a subgroup of  $V \rtimes G$ .)

(b) For  $L$  in  $A_G$  let  $\pi(L) = L \cap VH$ . Then it is easy to check that  $\rho_H \pi \rho_G^{-1}$  is just the restriction map  $M_G \rightarrow M_H$ ,  $\phi \rightarrow \phi_{VH/V}$ . Thus  $\pi$  is affine.

(c) Define  $\alpha: V \rightarrow M$  by  $\alpha(v)(a) = v^a - v$ . Then  $\ker \alpha = C_V(G)$  and  $\alpha(V) = \rho_G(I_G)$  is the set of inner derivations. In particular  $H^1(G, V) = M/\alpha(V) \cong A_G/I_G$  and (c) holds.  $\square$

*Proof of Theorem 2.* Let  $L \in \mathcal{L}$ . By (i) we may view  $V \rtimes L$  as a subgroup of  $H$  and by part (a) of the Lemma,  $A_L$  is an affine  $D$ -module and by (ii), (iii) and part (c) of the Lemma,  $A_L$  is finite dimensional. For  $L$  and  $K$  in  $\mathcal{L}$  with  $L \leq K$  let  $\pi_{K,L}$  be the affine map defined in part (b) of the Lemma. We claim that the inverse limit of  $(\pi_{K,L})_{L \leq K}$  is not empty. Note that finite dimensional affine  $D$ -modules fulfill the descending chain condition on affine subspaces and so a set of affine subspaces whose intersection is empty has a finite subset whose intersection is empty. Moreover, images and inverse images of affine subspaces under affine maps are affine. Now the proof in [KW, 1K1] that inverse limits of non-empty finite sets are not empty carries over word for word, except that “subset” has to be replaced by “affine subspace”. Let  $(C_L)_{L \in \Lambda}$  be an element in the inverse limit. Then  $\bigcup \{C_L | L \in \mathcal{L}\}$  is a complement to  $V$  in  $H$  and Theorem 2 is proved.  $\square$

*Proof of Theorem 3.* For  $X \leq V^*$  let  $X^\perp = \{v \in V | x(v) = 0 \text{ for all } x \in X\}$ . We will first prove that:

$$(*) \quad \text{For all } K \leq G, [V^*, K]^\perp = C_V(K).$$

Indeed, let  $x \in V^*$ ,  $k \in K$  and  $v \in V$ . Then

$$[x, k](v) = (x^k - x)(v) = x^k(v) - x(v) = x(v^{k^{-1}}) - x(v) = x([v, k^{-1}]).$$

It follows that  $v \in [V^*, K]^\perp$  if and only if  $[v, K] \leq V^{\perp} = 0$  and so if and only if  $v \in C_V(K)$ .

Let  $H \in \mathcal{L}$ . Define a map  $a_H: W \rightarrow [V^*, H]^*$  by  $a_H(w)(x) = x(u)$  where  $x \in [V^*, H]$ ,  $w \in W$  and  $u \in V$  with  $w \in u + C_W(H)$ . Note that by (\*) this definition does not depend on the choice of  $u$ . If  $K \leq H$  with  $K \in \mathcal{L}$ , then  $C_W(H) \leq C_W(K)$  and so  $w \in u + C_W(K)$  and  $a_H(w)(x) = a_K(w)(x)$  for all  $x \in [V^*, K]$ . Define  $a: W \rightarrow [V^*, G]^*$  by  $a(w)(x) = a_H(w)(x)$  whenever  $w \in W$ ,  $x \in [V^*, G]$  and  $H \in \mathcal{L}$  with  $x \in [V^*, H]$ . By the preceding observation and since  $\mathcal{L}$  is a local system this definition does not depend on the choice of  $H$ . Let  $w \in W$  with  $a(w) = 0$ . Then  $a_H(w) = 0$  for all  $H \in \mathcal{L}$  and so  $u \in [V^*, H]^\perp$ , where  $u$  is as above. By (\*),  $u \in C_V(H)$  and so  $w \in C_W(H)$  for all  $H \in \mathcal{L}$ . Thus  $\ker a = C_W(G)$ . It remains to show that  $a$  is a  $DG$ -homomorphism. Clearly  $a$  is  $D$ -linear. Let  $w, x, u$  and  $H$  be as above and  $g \in G$ . We may assume without loss that  $g \in H$ . Then  $w^g \in u^g + C_W(H)$  and so

$$\begin{aligned} a(w^g)(x) &= a_H(w^g)(x) = x(u^g) = x^{g^{-1}}(u) \\ &= a_H(w)(x^{g^{-1}}) = a(w)(x^{g^{-1}}) = a(w)^g(x). \end{aligned}$$

Thus  $a(w^g) = a(w)^g$  and  $a$  is a  $DG$ -homomorphism, completing the proof of Theorem 3.  $\square$

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN  
48824

*E-mail address:* `meier@math.msu.edu`