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PROPERTIES OF SUBGENERATORS OF C−REGULARIZED SEMIGROUPS

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Abstract. We introduce two operations $\wedge,\vee$ in the set $\mathcal{G}$ of subgenerators of a given C - regularized semigroup and prove that $\mathcal{G}$ is a complete partially ordered lattice with respect to $\wedge,\vee$ and the operator inclusion $\subseteq$. Also presented are some other properties and examples for $\mathcal{G}$.

1. Introduction

In dealing with the many physical problems that may be modeled as an abstract Cauchy problem

$$\text{(ACP)} \qquad \frac{d}{dt}u(t,x) = Au(t,x) \ (t \geq 0), u(0,x) = x,$$

where A is a closed linear operator on a Banach space X, and $u(\cdot,x) \in \mathbf{C}([0,\infty),X)$, well-posedness correponds to A generating a strongly continuous semigroup. When (ACP) is not well-posed, a useful concept for dealing with it is a C−regularized semigroup (Definition 1.1). When A generates a C−regularized semigroup, then (ACP) has a unique mild solution, for all x in the image of C, a unique strong solution, for all $x \in C(D(A))$, and $u(t,x_n) \longrightarrow u(t,x)$, uniformly for t in every compact subset of $[0,\infty)$, whenever $C^{-1}x_n \longrightarrow C^{-1}x$.

However, in order that (ACP) have all these solutions, it is not necessary that A generates a C−regularized semigroup; it is only necessary that A be a subgenerator (Definition 1.2) of a C−regularized semigroup. This was first observed only very recently, in [7, Counter example 0.2]. In fact, (ACP) has a unique mild solution for all $x \in \mathrm{Im}(C)$ if and only if A is a subgenerator of a C−regularized semigroup [7, Theorem 3.3].

Thus from the point of view of applications, there is no difference between a generator and a subgenerator. In practice, verifying that A is a subgenerator of a C−regularized semigroup is much easier than showing that A itself is the generator. Hence, the subgenerators become the objects of interest. Unfortunately, up to now the properties of the set of subgenerators of a C−regularized semigroup are not clear. This paper attempts to study these and presents several results and examples.

Throughout X is a Banach space and $L(X)$ is the algebra of all bounded linear operators on X. Let $C \in L(X)$.

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Definition 1.1 ([3], [4], [6], [7], [8]). The strongly continuous family of operators

$$\{W(t)\}_{t\geq 0} \subset L(X)$$

is a $C-$regularized semigroup if it satisfies:

(1) $W(0) = C$, and

(2) $W(t)W(s) = CW(t+s)$, for all $t, s \geq 0$.

$\{W(t)\}_{t\geq 0}$ is nondegenerate if $W(t)x \equiv 0$, for all $t \geq 0$, implies $x = 0$.

In [7, Proposition 2.2] it is shown that $\{W(t)\}_{t\geq 0}$ is nondegenerate if and only if C is injective. In this paper, we assume that $\{W(t)\}_{t\geq 0}$ is nondegenerate.

Definition 1.2 ([7], [12]). Assume B is closed. We say that B is a subgenerator of the $C-$regularized semigroup $\{W(t)\}_{t\geq 0}$ if

(1) $W(t)B \subseteq BW(t)$ for all $t \geq 0$, and

(2) $\int_0^t W(s)xds \in D(B)$ and $B\int_0^t W(s)xds = W(t)x - Cx \quad \forall x \in X, t \geq 0$.

We also say that B has a $C-$regularized semigroup $\{W(t)\}_{t\geq 0}$ or $\{W(t)\}_{t\geq 0}$ is a $C-$regularized semigroup for B.

For convenience, we will use the term subgenerator.

Generally, subgenerators of a given $C-$regularized semigroup are not unique (see Examples 2.13, 2.14). However, it is shown in [7, Proposition 2.9] that a $C-$regularized semigroup is uniquely determined by one of its subgenerators.

Definition 1.3 ([7]). Assume $\{W(t)\}_{t\geq 0}$ is a $C-$regularized semigroup. Let $D(\tilde{A})$ be the set of all $x \in X$ such that there exists $y \in X$ satisfying

$$W(t)x - Cx = \int_0^t W(s)yds \quad \forall t \geq 0.$$

Then $\tilde{A}x = y$. $\tilde{A}$ is called the generator of $\{W(t)\}_{t\geq 0}$.

An analogous definition of the generator of an integrated semigroup appears in [1], [14]. Moreover, it has been proved in [7, Proposition 2.6] that

(1) if $\tilde{A}$ is the generator of $\{W(t)\}_{t\geq 0}$ then $\tilde{A}$ is a subgenerator of $\{W(t)\}_{t\geq 0}$;

(2) $\tilde{A}x = C^{-1}\lim_{t\longrightarrow 0^+}\frac{1}{t}(W(t)x - Cx)$, with maximal domain.

2. Properties of subgenerators

In this section, we are devoted to the study of properties of subgenerators. Let $\mathcal{G}$ be the set of all subgenerators of the $C-$regularized semigroup $\{W(t)\}_{t\geq 0}$.

Definition 2.1 ([12]). Let A be the operator defined by

$$D(A) = \{\sum_{k=1}^m \int_0^{t_k} W(s)x_kds : x_k \in X, t_k \geq 0, k = 1, ..., m\};$$

$$A[\sum_{k=1}^m \int_0^{t_k} W(s)x_kds] = \sum_{k=1}^m [W(t_k)x_k - Cx_k].$$

Proposition 2.2 ([7], [12]). *Assume $\{W(t)\}_{t\geq 0}$ is a $C-$regularized semigroup. Then*

(1) *A in Definition 2.1 is well-defined and closable;*

(2) *$\bar{A}$, the closure of A, and $\tilde{A}$ are respectively the smallest and largest elements in $\mathcal{G}$, that is, every $B \in \mathcal{G}$ satisfies $\bar{A} \subseteq B \subseteq \tilde{A}$;*

(3) *for every $B \in \mathcal{G}, C^{-1}BC = \tilde{A}$.*

The following lemma is clear.

Lemma 2.3. *For every* $T \in L(X)$ *and every closable* S, $TS \subseteq ST$ *implies* $T\bar{S} \subseteq \bar{S}T$, *where* $\bar{S}$ *is the closure of* S.

Lemma 2.4. *For any* $B_1, B_2 \in \mathcal{G}$, *the following hold:*
(1) *if* $x \in D(B_1) \cap D(B_2)$ *then* $B_1 x = B_2 x$;
(2) *let* B *be the operator*

$$D(B) = D(B_1) \cap D(B_2), Bx = B_1 x \quad \forall x \in D(B).$$

Then B *is closed and* $B \in \mathcal{G}$.

Proof. (1) is obvious. As for (2), from Proposition 2.2(2), $\bar{A} \subseteq B \subseteq \tilde{A}$. This, together with the obvious inclusion $W(t)B \subseteq BW(t)$ for all $t \geq 0$, gives $B \in \mathcal{G}$. □

Lemma 2.5. *For any* $B_1, B_2 \in \mathcal{G}$, *let* B^0 *be the operator*

$$D(B^0) = \operatorname{span}[D(B_1) \cup D(B_2)];$$

$$B^0(a_1 x_1 + a_2 x_2) = a_1 B_1 x_1 + a_2 B_2 x_2 \quad \forall x_i \in D(B_i), \quad a_i \in \mathbf{C}, i = 1, 2.$$

Then B^0 *is closable and its closure* $\tilde{B} \in \mathcal{G}$.

Proof. B^0 is clearly the restriction of $\tilde{A}$ to $\operatorname{span}[D(B_1) \cup D(B_2)]$, it is well-defined and closable. The inclusions $\bar{A} \subseteq B^0 \subseteq \tilde{A}$ imply $\bar{A} \subseteq \tilde{B} \subseteq \tilde{A}$. From Lemma 2.3, $W(t)\tilde{B} \subseteq \tilde{B}W(t)$ for all $t \geq 0$. Hence $\tilde{B} \in \mathcal{G}$. □

Definition 2.6. For any $B_1, B_2 \in \mathcal{G}$, define $B = B_1 \wedge B_2, \tilde{B} = B_1 \vee B_2$, where $B, \tilde{B}$ are operators defined in Lemmas 2.4 and 2.5, respectively.

Proposition 2.7. *With respect to the operations* $\wedge, \vee$ *and the operator inclusion* $\subseteq, \mathcal{G}$ *is a complete partially ordered lattice.*

Proof. It suffices to claim that $\mathcal{G}$ is complete with respect to $\wedge, \vee$, since the fact that $\mathcal{G}$ is partially ordered with respect to the operator inclusion $\subseteq$ is clear.

For a family $\{B_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{G}$, define B to be the operator

$$D(B) = \bigcap_{\alpha \in \mathcal{A}} D(B_\alpha);$$

$$Bx = B_\alpha x \quad \forall x \in D(B) \text{ and } \alpha \in \mathcal{A}.$$

Define $\tilde{B}$ to be the closure of the following operator:

$$D(B^0) = \operatorname{span} \bigcup_{\alpha \in \mathcal{A}} D(B_\alpha);$$

$$B^0(a_1 x_{\alpha_1} + ... + a_k x_{\alpha_k}) = a_1 B_{\alpha_1} x_{\alpha_1} + ... + a_k B_{\alpha_k} x_{\alpha_k},$$

where $x_{\alpha_j} \in D(B_{\alpha_j}), a_j \in \mathbf{C}$ for $j = 1, 2, ..., k$. Then $B, \tilde{B} \in \mathcal{G}$ are the lower bound and upper bound of $\{B_\alpha\}_{\alpha \in \mathcal{A}}$, respectively. $\mathcal{G}$ is thus a partially ordered lattice. □

Proposition 2.8. *Assume* $\operatorname{Im}(C)$ *is dense in* X. *Then the following hold.*
(1) $\bar{A}$ *equals the closure of* $\tilde{A}$ *restricted to* $C(D(\tilde{A}))$.
(2) $\mathcal{G}$ *is a singleton if and only if* $C(D(\tilde{A}))$ *is a core for* $\tilde{A}$.

Proof. (1) follows from [7, Proposition 2.6(3) and Theorem 3.3(h)].

(2). From (1), $\mathcal{G}$ is a singleton if and only if $\tilde{A}$ is the closure of itself restricted to $C(D(\tilde{A}))$ if and only if $C(D(\tilde{A}))$ is a core for $\tilde{A}$. ☐

Corollary 2.9. *If $\tilde{A}$ is densely defined and $C(D(\tilde{A}))$ is a core for $\tilde{A}$, then $\mathcal{G}$ consists of only the element $\tilde{A}$.*

Proof. Since $C(D(\tilde{A}))$ is a core for $\tilde{A}$, it is dense in $D(\tilde{A})$. By the density of $D(\tilde{A})$, $C(D(\tilde{A}))$ is dense in X. Now the corollary follows from Proposition 2.8. ☐

Proposition 2.10. *$\mathcal{G}$ is totally ordered if and only if $\mathcal{G}$ contains at most two elements.*

Proof. "Only if". Suppose that $\mathcal{G}$ contains at least three elements A_1, A_2, A_3 satisfying

$$A_1 \subsetneq A_2 \subsetneq A_3.$$

Let $x_0 \in D(A_3) \setminus D(A_2)$. Define

$$D(A_2') = D(A_1) + \{ax_0\}, \quad a \in \mathbf{C};$$

$$A_2'(y + ax_0) = A_1 y + aA_3 x_0 \quad \forall y \in D(A_1), a \in \mathbf{C}.$$

Then A_2' is well-defined. We now claim that A_2' is closed. Assume $y_n + a_n x_0 \longrightarrow x$ and $A_2'(y_n + a_n x_0) \longrightarrow z$, as $n \longrightarrow \infty$. Then $\{a_n\}$ is bounded. Otherwise we may assume $a_n \longrightarrow \infty$. Then $(y_n + a_n x_0)/a_n \longrightarrow 0$, hence $y_n/a_n \longrightarrow -x_0$. Since

$$A_1(y_n/a_n) = A_2'(y_n + a_n x_0)/a_n - A_3 x_0 \longrightarrow -A_3 x_0$$

and A_1 is closed, we have $x_0 \in D(A_1)$, contradicting the fact that $x_0 \notin D(A_1)$. Thus we may assume $a_n \longrightarrow a_0$, as $n \longrightarrow \infty$. From

$$y_n \longrightarrow x - a_0 x_0, \quad \text{and}$$

$$A_1 y_n = A_2'(y_n + a_n x_0) - a_n A_3 x_0 \longrightarrow z - a_0 A_3 x_0,$$

we have $x - a_0 x_0 \in D(A_1)$ and

$$A_1(x - a_0 x_0) = z - a_0 A_3 x_0.$$

This implies

$$x = (x - a_0 x_0) + a_0 x_0 \in D(A_2'), \quad \text{and}$$

$$A_2' x = A_1(x - a_0 x_0) + a_0 A_3 x_0 = z.$$

A_2' is closed.

Next, we prove that

$$W(t)A_2' \subseteq A_2' W(t) \quad \forall t \geq 0. \tag{2.1}$$

For $x \in D(A_3)$, differentiate both sides of

$$A_1 \int_0^t W(s)x\,ds = \int_0^t W(s)A_3 x\,ds$$

to obtain

$$A_1 W(t)x = W(t)A_3 x \tag{2.2}$$

by the closedness of A_1. For the previous y and x_0, (2.2) implies

$$W(t)A_2'(y+ax_0) = W(t)A_3(y+ax_0) = A_1W(t)(y+ax_0) = A_2'W(t)(y+ax_0),$$

proving (2.1). Hence $A_2' \in \mathcal{G}$. Clearly, A_2, A_2', are not comparable with respect to the operator inclusion, contradicting the hypotheses on $\mathcal{G}$.

"If" is clear. □

Proposition 2.11. *If $\mathcal{G}$ is finite then there exists $n \in N \cup \{0\}$ such that the cardinality of $\mathcal{G}$ is 2^n.*

Proof. Since $\mathcal{G}$ is finite, the codimension of $D(\bar{A})$ in $D(\tilde{A})$ is finite. Assume $\bar{A} \subsetneq \tilde{A}$. Then there exist $n \in N$ and linearly independent elements $x_1, x_2, ..., x_n$ in $D(\tilde{A}) \setminus D(\bar{A})$ such that

$$D(\tilde{A}) = D(\bar{A}) \oplus \operatorname{span}\{x_1, x_2, ..., x_n\}, \tag{2.3}$$

where "$\oplus$" is the algebraic direct sum. For any subset $\{x_{n_1}, ..., x_{n_k}\}(1 \le k \le n)$ of $\{x_1, ..., x_n\}$, define

$$\begin{cases} D(B) & = D(\bar{A}) \oplus \operatorname{span}\{x_{n_1}, ..., x_{n_k}\}, \text{ and} \\ B(y+x) & = \bar{A}y + \tilde{A}x, \end{cases} \tag{2.4}$$

where $y \in D(\bar{A}), x = a_1x_{n_1} + ... + a_kx_{n_k}$ for some $a_j \in \mathbf{C}(1 \le j \le k)$. As with the argument for the closedness of A_2' in Proposition 2.10, it is easy to show that B is closed by induction. Moreover,

$$W(t)B \subseteq BW(t);$$

$$B\int_0^t W(s)xds = W(t)x - Cx, \quad \forall x \in X.$$

B is a subgenerator of $\{W(t)\}_{t \ge 0}$.

Now assume $B \in \mathcal{G}$ and $\bar{A} \subsetneq B$. From (2.3) there exists $\{x_{n_1}, ..., x_{n_k}\} \subseteq \{x_1, ..., x_n\}$ $(1 \le k \le n)$ such that every $z \in D(B)$ has the decomposition

$$z = y + x, \tag{2.5}$$

where $y \in D(\bar{A}), x = a_1x_{n_1} + ... + a_kx_{n_k}$ for some $a_j \in \mathbf{C}(1 \le j \le k)$ and k is the minimal positive integer such that (2.5) holds for all $z \in D(B)$. Then

$$Bz = By + Bx = \bar{A}y + \tilde{A}x.$$

Hence every subset of $\{x_1, ..., x_n\}$ corresponds to a unique element B in $\mathcal{G}$ defined as in (2.4) and *vice versa.* In particular, the empty set corresponds to $\bar{A}$ and $\{x_1, ..., x_n\}$ itself corresponds to $\tilde{A}$. Since the cardinality of the collection of all subsets of $\{x_1, ..., x_n\}$ is 2^n, that of $\mathcal{G}$ is also.

The following examples show that the set $\mathcal{G}$ of subgenerators of a given $C-$regularized semigroup may contain 2^n elements for every $n \in N \cup \{0\}$ or even infinitely many elements.

Example 2.12 ([8]). Let μ be the Lebesgue measure on $\mathbf{C}$. Define the operator $\tilde{A}$ on $L^2(\mathbf{C}, \mu)$:

$$(\tilde{A}f)(z) = zf(z); \quad D(\tilde{A}) = \{f | f(z), zf(z) \in L^2(C, \mu)\}.$$

It is known that $\tilde{A}$ generates the $\exp(-|\tilde{A}|^2)-$regularized semigroup

$$(W(t)f)(z) = e^{-|z|^2}e^{tz}f(z).$$

We now prove that the set $\mathcal{G}$ of $\{W(t)\}_{t\geq 0}$ consists of only the element $\tilde{A}$.

Since the set of those f's in $L^2(\mathbf{C},\mu)$ with compact support is a core for $\tilde{A}$, for every $f \in D(\tilde{A})$, there exists a sequence $\{f_n\}$ in $L^2(\mathbf{C},\mu)$ with compact support such that

$$\int_{\mathbf{C}} |f_n(z) - f(z)|^2 d\mu \longrightarrow 0, \int_{\mathbf{C}} |z|^2|f_n(z) - f(z)|^2 d\mu \longrightarrow 0, \quad \text{as} \ \ n \longrightarrow \infty. \tag{2.6}$$

Write $g_n(z) = f_n(z)e^{|z|^2}$. Then $g_n \in D(\tilde{A})$. Introduce $f_n(z) = g_n(z)e^{-|z|^2}$ into (2.6) to conclude that $e^{-|\tilde{A}|^2}(D(\tilde{A}))$ is a core for $\tilde{A}$. Since $\tilde{A}$ is densely defined, Corollary 2.9 gives the conclusion.

Example 2.13. Let $X = l^2$. For $x = (\xi_1, \xi_2, ...) \in X$, define

$$W(t)x = e^t(0, \xi_1, \xi_2, ...) \ \text{ for } t \geq 0.$$

Then $\{W(t)\}$ is a $C-$regularized semigroup with

$$C : x = (\xi_1, \xi_2, ...) \longrightarrow (0, \xi_1, \xi_2, ...).$$

It is easy to see that $\{I, I_0\} = \mathcal{G}$, where I is the identity on X and I_0 is the identity on $X_0 \equiv \operatorname{Im}(C)$.

More generally, let $n \in N$ and define

$$W(t)x = e^t(\underbrace{0, ..., 0,}_{n \text{ folds}}\xi_1, \xi_2, ...) \ \ \forall x = (\xi_1, \xi_2, ...) \in X.$$

Then $\{W(t)\}_{t\geq 0}$ is a $C-$regularized semigroup with

$$C : x = (\xi_1, \xi_2, ...) \longrightarrow (\underbrace{0, ..., 0,}_{n \text{ folds}}\xi_1, \xi_2, ...),$$

and $\mathcal{G}$ contains 2^n elements.

Example 2.13 shows that there exists a $C-$regularized semigroup so that even if $\tilde{A}(= I)$ is bounded, the set $\mathcal{G}$ of subgenerators may contain more than one element. This is because $C(D(\tilde{A}))(= I_0(D(\tilde{A}))$ is not a core for $\tilde{A}$. □

Example 2.14 ([7]). Let $G \equiv \frac{d}{dx}$, on $X \equiv L^\infty(R)$, with maximal domain. Let A be the restriction of G to $D(G^2)$, the domain of G^2; that is,

$$D(A) \equiv D(G^2), Ax \equiv Gx \ \ \forall x \in D(A).$$

Let $\bar{A}$ be the closure of A. Let $C \equiv (1-G)^{-2}$, and define a $C-$regularized semigroup $\{W(t)\}_{t\geq 0}$ by

$$[W(t)f](s) = (Cf)(t+s) \ \ (t \geq 0, s \in R).$$

Then G is the generator of $\{W(t)\}_{t\geq 0}$. The domain of $\bar{A}$ equals the graph closure of the domain of G^2, which may be shown to equal $(1-G)^{-1}(BUC(R))$, which does not equal $D(G) = (1-G)^{-1}(L^\infty(R))$, where BUC(R) is the space of all bounded uniformly continuous functions on R.

Before proceeding further, we define one more regularized semigroup $\{W_1(t)\}_{t\geq 0}$ by (see [7, Example 2.11])

$$[W_1(t)f](s) = [(I-G)^{-1}f](t+s) \quad (t \geq 0, s \in R).$$

Then G is also the generator of $\{W_1(t)\}_{t\geq 0}$ and for any $f \in X$,

$$\frac{1}{t}\int_s^{t+s}[(I-G)^{-1}f](r)dr = \frac{1}{t}\int_0^t [W_1(r)f](s)dr$$

$$\longrightarrow [W_1(0)f](s) = [(I-G)^{-1}f](s), \quad \text{as} \;\; t \longrightarrow 0+, \tag{2.7}$$

in X.

We now prove that $\bar{A}$ is the smallest element in $\mathcal{G}$. Let $A' \in \mathcal{G}$. From the boundedness of $G(I-G)^{-1}$, $A'(I-G)^{-2} = G(I-G)^{-2}$ and (2.7),

$$A'\{C[\frac{1}{t}\int_s^{t+s} f(r)dr]\} = G(I-G)^{-1}[\frac{1}{t}\int_s^{t+s}[(I-G)^{-1}f](r)dr$$

$$\longrightarrow [G(I-G)^{-2}f](s), \quad \text{as} \;\; t \longrightarrow 0+,$$

in X. (2.7) also implies that

$$C[\frac{1}{t}\int_s^{t+s} f(r)dr] \longrightarrow (Cf)(s), \quad \text{as} \;\; t \longrightarrow 0+,$$

in X. Then $Cf \in D(A')$ for any $f \in X$. This means that $\mathrm{Im}(C) \subseteq D(A')$, or equivalently, $D(A) \subseteq D(A')$. Hence $\bar{A} \subseteq A'$. $\bar{A}$ is thus the smallest element in $\mathcal{G}$.

Let $x_0 \in D(G) \setminus D(\bar{A})$ and define

$$A_1(y + ax_0) = \bar{A}y + aGx_0 \quad \forall y \in D(\bar{A}), a \in \mathbf{C}.$$

From the proof of Proposition 2.10, A_1 is in $\mathcal{G}$. Since there are infinitely many linearly independent choices of $x_0, \mathcal{G}$ contains infinitely many elements.

3. Remarks

In this last section, we present several remarks about the $C-$resolvent sets and eigenvalues of subgenerators of a given $C-$regularized semigroup.

Definition 3.1 ([6], [7]). Let A be closed. The complex number λ is in $\rho_c(A)$, the $C-$resolvent set of A, if $\lambda - A$ is injective and $\mathrm{Im}(C) \subseteq \mathrm{Im}(\lambda - A)$.

Lemma 3.2. *Let $\lambda \in \mathbf{C}, x \in D(\tilde{A})$, where $\tilde{A}$ is the generator of the $C-$regularized semigroup $\{W(t)\}_{t\geq 0}$. Then $(\lambda - \tilde{A})x = 0$ if and only if $(\lambda - B)Cx = 0$ for every $B \in \mathcal{G}$.*

Proof. Since $\tilde{A} = C^{-1}BC$, $\tilde{A}x = \lambda x$ if and only if $C^{-1}BCx = \lambda x$ if and only if $BCx = \lambda Cx$. ☐

Proposition 3.3. *The following are true.*

(1) *All elements in $\mathcal{G}$ have the same eigenvalues.*

(2) *Corresponding to the same eigenvalue λ, x is an eigenvector of $\tilde{A}$ if and only if Cx is an eigenvector of B for every $B \in \mathcal{G}$.*

(3) *For any $B_1, B_2 \in \mathcal{G}$, if $B_1 \subseteq B_2$ then $\rho_c(B_1) \subseteq \rho_c(B_2)$.*

Proof. (1),(2) are consequences of Lemma 3.2.

(3) Assume $\lambda \in \rho_c(B_1)$. Then $(\lambda - B_1)$ is injective, so $(\lambda - B_2)$ is also by (1). Since $\mathrm{Im}(C) \subseteq \mathrm{Im}(\lambda - B_1) \subseteq \mathrm{Im}(\lambda - B_2)$, we have $\lambda \in \rho_c(B_2)$. ☐

It is easy to verify that all results presented in this paper remain valid for sequentially complete locally convex spaces, which appear in [11], [12], and it is also easy to generalize all results obtained in this paper to the case of n-times integrated $C-$regularized semigroups (see [12]).

Open questions. The following were proposed by the referee.

(1) When $\mathrm{Im}(C)$ is dense in X, are there examples where $\mathcal{G}$ consists of more than one element? Or equivalently, when $\mathrm{Im}(C)$ is dense in X, are there examples where $C(D(\tilde{A}))$ is not a core for the generator $\tilde{A}$? (See Proposition 2.8(2).)

(2) Proposition 2.8(1) proves that when $\mathrm{Im}(C)$ is dense in X, then the minimal element of $\mathcal{G}$ equals the closure of $\tilde{A}$ restricted to $C(D(\tilde{A}))$. Does this remain true in general ?

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