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A CRITICAL METRIC FOR THE L^2-NORM OF THE CURVATURE TENSOR ON S^3

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ABSTRACT. The L^2-norm of the curvature tensor

$$\mathcal{R}(g) = \frac{1}{(\mathrm{Vol}(M))^{\frac{n-4}{n}}} \int_M |R|^2 \, dvol_g$$

defines a Riemannian functional on the space of metrics. This work exhibits a metric on S^3 which is of Berger type but not of constant ricci curvature, and yet is critical for $\mathcal{R}$.

0. INTRODUCTION

The purpose of this article is to exhibit, as the title suggests, a metric of non-constant ricci curvature on S^3 which is critical for the L^2-norm of the curvature tensor, a Riemannian functional that we denote by the letter $\mathcal{R}$.

In dimension three and four metrics of constant ricci curvature are critical points of $\mathcal{R}$. It is natural to ask whether the converse is true. In dimension four the question is quickly answered negatively by taking the product of a surface of curvature -1 with the standard two dimensional sphere. In dimension three a counterexample may be found among the set of left invariant metrics on $\mathbf{SU}(2)$.

As it happens the principle of symmetric criticality allows us to see the L^2 norm of the curvature tensor as a functional on the set of inner products on $\mathbf{su}(2)$. Moreover this functional is invariant under the automorphism group of $\mathbf{su}(2)$. We shall describe a one-parameter family $\{g_t\}$ of inner products which form the invariant set of a subgroup of the automorphism group. The value of $\mathcal{R}$ on the family $\{g_t\}$ is then given by a fourth degree polynomial in t. By the principle of symmetric criticality, the values which are critical for this polynomial are precisely the "instants" for which g_t is $\mathcal{R}$-critical. As we'll see, one of the critical metrics thus obtained is of non-constant ricci curvature.

The above counterexample was initially found in the author's classification of homogeneous, $\mathcal{R}$-critical metrics in dimension three and four (see [2]). This classification has the virtue of showing that, aside from the above counterexample, all other instances of homogeneous $\mathcal{R}$-critical metrics on compact three-manifolds are of constant ricci curvature. Nevertheless we believe this approach to be preferable, at least in two respects. Indeed, metrics of constant ricci curvature are the critical points of another Riemannian functional, namely, the integral of the scalar

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curvature $\mathcal{S}$. The principle of symmetric criticality allows us to restrict $\mathcal{S}$, or for that matter any other Riemannian functional, to the family of inner products $\{g_t\}$. But this time we obtain a degree two polynomial and thus only one critical point, i.e., the round sphere. This is the phenomenon at the heart of the existence of our counterexample. A higher degree Riemannian functional would yield a higher degree polynomial, and thus more critical points. We also believe that, while a classification of homogeneous $\mathcal{R}$-critical metrics in high dimension seems unwieldy, this approach could be used to fabricate higher dimensional examples of critical metrics.

The paper is divided in three sections. The first is a reminder of the necessary differential geometry and Lie theory needed. The second is the computation yielding our counterexample. The third establishes how an $\mathcal{R}$-critical, left invariant metric on $\mathbf{su}(2)$ is of Berger type, i.e., is obtained by shrinking uniformly the fibers of the Hopf fibration by a factor of $\frac{2}{\sqrt{11}}$.

1. The $\mathcal{R}$-functional on the space of left invariant metrics on $\mathbf{SU}(2)$

Let M be a compact, differentiable manifold of dimension n. Let g be a Riemannian metric on M, and R its curvature tensor. We define the L^2 norm of the curvature tensor to be

$$\mathcal{R}(g) = \frac{1}{(\mathrm{Vol}(M))^{\frac{n-4}{n}}} \int_M |R|^2 \mathrm{d}\mu.$$

Here $\mathrm{Vol}(M)$ is the volume of M, and $\mathrm{d}\mu$ is the volume element induced by g.

The goal of this section is to evaluate $\mathcal{R}$ on the space of left invariant metrics of $\mathbf{SU}(2)$. For this matter we recall a number of facts on left invariant metrics beautifully explained in [3].

A left invariant metric on a Lie group G is equivalent to an inner product on its Lie algebra $\mathbf{g}$. If the Lie algebra is unimodular and three dimensional, the relationship between the commuting relations and a given inner product takes a particularly simple form. Indeed, given any inner product on $\mathbf{g}$ there exists a canonical basis of orthonormal vectors $\{e_1, e_2, e_3\}$ for which the commutators are written as

$$\begin{aligned}[e_1, e_2] &= \lambda_3 e_3,\\ [e_2, e_3] &= \lambda_1 e_1,\\ [e_3, e_1] &= \lambda_2 e_2.\end{aligned}$$

Define $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. In the basis $\{e_1, e_2, e_3\}$ the ricci tensor r is diagonal with the following eigenvalues:

$$\begin{aligned}r(e_1, e_1) &= 2\mu_2\mu_3,\\ r(e_2, e_2) &= 2\mu_1\mu_3,\\ r(e_3, e_3) &= 2\mu_1\mu_2.\end{aligned}$$

Since we are in dimension three, the curvature tensor is fully expressible in terms of the ricci tensor. Whence

$$|R|^2 = 2|r|^2 - \frac{1}{2}s^2,$$

where s denotes the scalar curvature. Thus, if need be, we can express the norm of the curvature tensor in terms of the structure constants $\lambda_1, \lambda_2, \lambda_3$.

Computing the volume of G in terms of the constants λ_i is a subtler matter. We need an explicit description of the exponential map between the Lie algebra and the Lie group. This would ensure that we can concretely realise the Haar measure as a volume form on $\mathbf{g}$. Fortunately in the case of $\mathbf{SU}(2)$ the non-degeneracy of the Killing form yields a standard volume element on $\mathbf{su}(2)$. We denote the volume of $\mathbf{SU}(2)$ induced by the Killing form by the letter C. As such the volume induced by any other left invariant metric is a function of C and the structure constants.

Hence the L^2-norm of the curvature tensor of an $\mathbf{SU}(2)$-left invariant metric is fully expressible in terms of the structure constants and the volume induced by the Killing form.

2. The principle of symmetric criticality

Let us remind the reader of the principle of symmetric criticality as stated in [4].

Theorem 1. *Let G be a compact Lie group, X a smooth G-manifold, $\pi : Y \to X$ a smooth G-fiber bundle over X, and M a Banach manifold of sections of Y. Let G act on M by $(gU) = g(U(g^{-1}x))$ and let $f : M \to \mathbf{R}$ be a smooth G-invariant function on M. Then the set Σ of G-equivariant sections in M is a smooth submanifold of M, and if $U \in \Sigma$ is a critical point of $f|\Sigma$, then U is in fact a critical point of f.*

This has the obvious consequence that a left invariant metric on $\mathbf{SU}(2)$ needs only to be critical for the restriction of $\mathcal{R}$ to the space of the left invariant metric in order to be $\mathcal{R}$-critical over the space of all Riemannian metrics on $\mathbf{SU}(2)$.

Let e_1, e_2, e_3 be the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These form a basis for $\mathbf{su}(2)$ satisfying

$$[e_1, e_2] = 2e_3,$$
$$[e_2, e_3] = 2e_1,$$
$$[e_3, e_1] = 2e_2.$$

The Killing form B of $\mathbf{su}(2)$ then satisfies $B(e_i, e_j) = -2\delta_{ij}$. Since the group of automorphisms of $\mathbf{su}(2)$ preserves the Killing form, its identity component is contained in $\mathbf{SO}(3)$. In fact, it is $\mathbf{SO}(3)$ (see [1], p.132).

Define

$$g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{pmatrix}$$

to be an inner product on $\mathbf{su}(2)$ written in the basis $\{e_1, e_2, e_3\}$. The family $\{g_t\}$ is precisely the invariant set of $\mathbf{SO}(2)$ with its canonical embedding in the group of automorphisms. Hence by the principle of symmetric criticality an $\mathcal{R}$-critical metric is found as soon as $\frac{d}{dt}\mathcal{R}(g_t) = 0$.

In order to compute the value of $\mathcal{R}(g_t)$ observe that a canonical frame $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ for g_t is given by $\tilde{e_1} = e_1$, $\tilde{e_2} = e_2$, $\tilde{e_3} = \frac{1}{t}e_3$, and satisfies

$$[\tilde{e_1}, \tilde{e_2}] = 2t\tilde{e_3},$$
$$[\tilde{e_2}, \tilde{e_3}] = \frac{2}{t}\tilde{e_1},$$

$$[\tilde{e_3}, \tilde{e_1}] = \frac{2}{t}\tilde{e_2}.$$

So that $\mu_1 = t$, $\mu_2 = t$, $\mu_3 = \frac{2-t^2}{t}$.

Therefore

$$|r|^2 = (2\mu_1\mu_2)^2 + (2\mu_1\mu_3)^2 + (2\mu_2\mu_3)^2$$

yields

$$|r|^2 = 32 - 32t^2 + 12t^4;$$

similarly

$$s^2 = 64 - 32t^2 + 4t^4.$$

Since we're in dimension three $|R|^2 = 2|r|^2 - 1/2s^2$. Thus

$$|R|^2 = 32 - 48t^2 + 22t^4.$$

Let C be the volume of $\mathbf{SU}(2)$ induced by its Killing form. Then

$$\mathrm{Vol}_{g_t}(\mathbf{SU}(2)) = \frac{1}{2\sqrt{2}}Ct.$$

Hence

$$\begin{aligned}\mathcal{R}(g_t) &= \mathrm{Vol}_{g_t}(\mathbf{SU}(2))^{4/3}|R|^2\\ &= (\frac{C}{2\sqrt{2}})^{4/3}t^{4/3}(32 - 48t^2 + 22t^4).\end{aligned}$$

By a simple computation we see that $\mathcal{R}(g_t)$ is critical at $t^2 = 1$, and $t^2 = 4/11$. The first critical point corresponds to the round sphere, the second is described in the next section.

3. Geometric description of an $\mathcal{R}$-critical metric

Since the group $\mathbf{SU}(2)$ can be identified with the unit quaternions, i.e., with S^3, we may hope to find a natural realisation for the metric g_t when $t = \frac{2}{\sqrt{11}}$. Let us recall the details of this identification.

Let $(z, w) \in \mathbf{C}^2$ and (z_1, z_2, w_1, w_2) be the real coordinates of (z, w). The map $f : \mathbf{SU}(2) \to S^3$ defined by

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \to (z + \mathbf{j}w)$$

is a Lie group isomorphism for which the action of $\mathbf{SU}(2)$ on $\mathbf{C}^2$ is identified with the action of S^3 on $\mathbf{R}^4$ by quaternionic multiplication. Namely

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (z + \mathbf{j}w)(a + \mathbf{j}b) = za - \bar{w}b + \mathbf{j}(\bar{z}b + wa).$$

Moreover f sends the identity element of $\mathbf{SU}(2)$ to the vector $(1, 0, 0, 0)$. Thus $df_{\mathrm{id}} : \mathbf{su}(2) \to T_{(1,0,0,0)}S^3$ is an isomorphism sending

$$\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$$

to, respectively, $\{\frac{\partial}{\partial z_2}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\}$. Define $\{e_1, e_2, e_3\}$ to be the left invariant vector fields on S^3 corresponding to $\{\frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial z_2}\}$. We then retrieve the same commuting relations as in the previous section. Moreover, since $\mathbf{SU}(2) \subset \mathbf{SO}(4)$, the frame $\{e_1, e_2, e_3\}$ is orthonormal with respect to the round metric on S^3.

The Hopf fibration may be obtained by a right S^1 action on S^3. Namely $(z,w)e^{i\theta} \to (ze^{i\theta}, we^{i\theta})$. Clearly the vector field e_3 is tangent to these orbits. Hence the metric

$$g = e_1 \otimes e_1 + e_2 \otimes e_2 + \frac{4}{11} e_3 \otimes e_3$$

is of Berger type.

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