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A BOUND FOR THE NILPOTENCY OF A GROUP OF SELF HOMOTOPY EQUIVALENCES

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Abstract. Let $\mathcal{E}_\Omega(X)$ be the group of homotopy classes of self-homotopy equivalences of X such that $\Omega f \simeq 1d_{\Omega X}$. We prove that $\mathcal{E}_\Omega(X)$ is a nilpotent group and that $\operatorname{nil}\mathcal{E}_\Omega(X) \leq \operatorname{cat}(X) - 1$.

Given a pointed space X of the homotopy type of a CW-complex, let $\mathcal{E}(X)$ denote the group of based homotopy classes of self homotopy equivalences of X ([1] is an excellent survey on this object). A considerable amount of work has been dedicated to obtaining finiteness properties, not only of $\mathcal{E}(X)$, but also of certain interesting subgroups which preserve additional geometrical structure (see for example [2],[5],[6],[8]). This note goes in this direction: Let $\mathcal{E}_\Omega(X)$ be the kernel of the obvious map $\mathcal{E}(X) \to \mathcal{E}(\Omega X)$ (i.e. homotopy classes of equivalences $f\colon X \to X$ such that $\Omega f \simeq 1_{\Omega X}$) and, as usual, denote by $\operatorname{cat}(X)$ the Lusternik-Schnirelmann category of X. Then we prove:

Theorem. *If* $\operatorname{cat} X$ *is finite then* $\mathcal{E}_\Omega(X)$ *is a nilpotent group and nil* $\mathcal{E}_\Omega(X) \leq$ *cat*$(X) - 1$.

Remarks. (a) Observe that $\mathcal{E}_\Omega(X)$ is a subgroup of the group $\mathcal{E}_\#(X)$ consisting of homotopy classes of equivalences inducing the identity on the homotopy groups of X. Therefore it is known to be nilpotent for finite complexes in view of [4, Thm. B]. Note also that, in general, this inclusion is proper as is shown in the following example comunicated to us by F. Cohen: It is known [4, Cor. 1.3] that, given a prime $p \geq 3$ and $n \geq 1$, p^n is an exponent for S^{2n+1} at p. Therefore, if we consider ρ the p^n-th power map on the space $X = (\Omega^{2n-3}S^{2n+1}\langle 2n+1\rangle)_{(p)}$ and call $\sigma = 1+\rho$, it follows that $\pi_*(\sigma) = 1_{\pi_*(X)}$. On the other hand $\Omega(\rho)$ is essential [9, Thm. 1] and thus $\Omega(\sigma)$ cannot be homotopic to the identity.

(b) However, for rational spaces it is well known that $\mathcal{E}_\Omega(X) = \mathcal{E}_\#(X)$ since in this case ΩX has the homotopy type of a product of Eilenberg-Mac Lane spaces of type $(n_i, \mathbb{Q})$ in which the integers $\{n_i\}$ describe the degrees of a basis of $\pi_*(X)$. Hence, the theorem above could be seen as a generalization of [6, Thm. 1]

The rest of the paper is devoted to the proof of the theorem above. To simplify the notation we shall not distinguish between a homotopy class and a map which represents it. Also, equality of homotopy classes (or maps) will often mean homotopy between its representatives.

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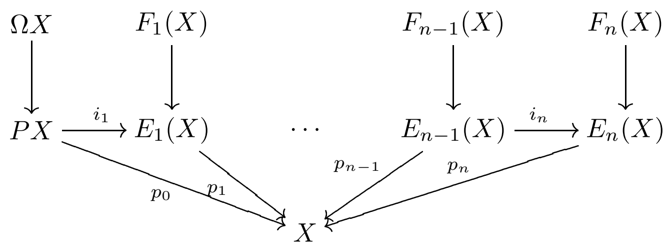
To start, let us recall the characterization of the LS category of a space X given in [7]. The n-th Ganea fibration of X, $F_n(X) \to E_n(X) \overset{p_n}{\to} X$, is defined by an inductive procedure in the following way: p_o is just the path fibration $\Omega X \to PX \overset{p_o}{\to} X$. Next consider C the homotopy cofibre of the inclusion $F_{n-1}(X) \to E_{n-1}(X)$ and extend p_{n-1} to a map $C \to X$. The associated fibration to this map $F_n(X) \to E_n(X) \overset{p_n}{\to} X$ is by definition the n-th Ganea fibration of X. $E_n(X)$ is called the n-th Ganea space for X. As a general picture we have:

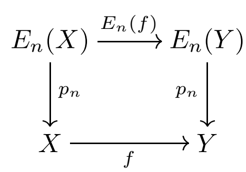


Then, we shall make use of the following facts:

(1) cat $X \leq n$ if and only if p_n admits a homotopy section.

(2) $F_n(X)$ has the homotopy type of the join of $n+1$ copies of ΩX.

(3) For each space X and each integer n, the fibration $E_n(X) \overset{p_n}{\to} X$ defines an augmented functor, that is to say, given $f\colon X \to Y$, there exists a (functorial) map $E_n(f)\colon E_n(X) \to E_n(Y)$ such that the following diagram commutes:



Next, given a map $f\colon X \to X$ representing an element on $\mathcal{E}_\Omega(X)$ define the length of f, $l(f)$, as the biggest integer n for which $fp_n = p_n$. Since $E_1(X)$ has the homotopy type of $\Sigma\Omega X$ and (up to homotopy) $p_1\colon \Sigma\Omega X \to X$ is the adjoint to the identity, $l(f)$ is at least 1. Also, observe that if $l(f) = n$, then $fp_m = f$ for any $m \leq n$. Next, define G_n as the subgroup of $\mathcal{E}_\Omega(X)$ consisting of equivalences of length at least n. Clearly $G_1 = \mathcal{E}_\Omega(X)$ and $G_{n+1} \subset G_n$.

Lemma. $[G_1, G_n] \subset G_{n+1}$.

Proof. First, recall [10] that given a cofibration sequence $Y \to Z \to C$, the coaction $\nu\colon C \to \Sigma Y \vee C$ induces a natural action of the group $[\Sigma Y, X]$ on $[C, X]$. Explicitly, given $\beta \in [\Sigma Y, X]$ and $\alpha \in [C, X]$, define $\alpha^\beta = (\beta, \alpha) \circ \nu$. The orbits of this action are precisely $i_*^{-1}(h)$, $h \in [Z, X]$, with $i_*\colon [C, X] \to [Z, X]$ induced by i. That is to say, given maps $\alpha_1, \alpha_2\colon C \to X$, $\alpha_1 i \sim \alpha_2 i$ if and only if there exists $\beta\colon \Sigma Y \to X$ such that $\alpha_1^\beta = \alpha_2$.

Note also that given $\gamma \in [X, W]$ and $\varphi \in [C, C]$, $\gamma\alpha^\beta = (\gamma\alpha)^{\gamma\beta}$ and $\alpha^\beta\varphi = (\alpha\varphi)^{\beta\Sigma\varphi}$, with $\Sigma\varphi \in [\Sigma Y, \Sigma Y]$ induced by φ by collapsing Z. We return to the proof of the lemma. Let $f, g\colon X \to X$ be maps satisfying $fp_1 = p_1$ and $gp_n = p_n$. We will prove that $fgp_{n+1} = gfp_{n+1}$. For that we shall apply the considerations above to the cofibration sequence $F_n(X) \to E_n(X) \overset{i_{n+1}}{\to} E_{n+1}(X)$. Since $gp_n = p_n$

and $p_n = p_{n+1} i_{n+1}$, there exists $h\colon \Sigma F_n(X) \to X$ such that $g p_{n+1} = p_{n+1}^h$. Observe that:

(i) Since h factors as the composite $\Sigma F_n(X) \xrightarrow{k} \Sigma\Omega X \xrightarrow{p_1} X$, we have $fh = f p_1 k = p_1 k = h$.

(ii) On the other hand, since $\Omega f = 1$, via (2), it follows that $F_n(f) = *^{n+1}\Omega f = 1$.

Finally we can write:

$$f g p_{n+1} = f p_{n+1}^h = (f p_{n+1})^{fh} \underset{\text{(i)}}{=} (f p_{n+1})^h \underset{\text{(ii)}}{=} (f p_{n+1})^{h \Sigma F_n(f)}$$
$$\underset{(3)}{=} (p_{n+1} E_{n+1}(f))^{h\Sigma F_n(f)} = p_{n+1}^h E_{n+1}(f) = g p_{n+1} E_{n+1}(f) = g f p_{n+1}. \quad \square$$

Proof of the theorem. Observe that if cat $X = m$ then $G_m = \{1\}$. Indeed, given $f \in G_n$ and in view of (1), $f = f p_m \sigma = p_m \sigma = 1$ with σ section of p_n. Hence, by lemma above we have a finite decreasing sequence of normal subgroups

$$\mathcal{E}_\Omega(X) = G_1 \supset G_2 \supset \ldots \supset G_m = \{1\}$$

in which $[G_1, G_n] \subset G_{n+1}$ and thus the theorem follows. $\square$

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