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UNIQUENESS OF POSITIVE SOLUTIONS FOR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. Sufficient conditions for the uniqueness of positive solutions of singular Sturm-Liouville boundary value problems

$$\text{(BVP)}\qquad \begin{cases} \text{(E)}\ (|u'|^{m-2}u')' + f(t,u,u') = 0, \quad \text{in } (\theta_1,\theta_2), m \geq 2,\\ \text{(BC)} \begin{cases} \alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) = 0,\\ \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i \geq 0$ and $\alpha_i^2 + \beta_i^2 \neq 0$ $(i = 1, 2)$, are established.

1. Introduction

In this paper, we are concerned with the uniqueness of positive solutions of boundary value problems for the nonlinear differential equation

$$\text{(E)}\qquad (|u'|^{m-2}u')' + f(t,u,u') = 0, \qquad \theta_1 < t < \theta_2,\ m \geq 2,$$

subject to one of the following sets of boundary conditions:

$$\text{(BC.1)}\qquad u(\theta_1) = \xi_1 \geq 0, \quad u'(\theta_2) = \xi_2 \geq 0,$$

$$\text{(BC.2)}\qquad u'(\theta_1) = \xi_1 \leq 0, \quad u(\theta_2) = \xi_2 \geq 0,$$

$$\text{(BC.3)}\qquad u(\theta_1) = \xi_1 \geq 0, \quad u(\theta_2) = \xi_2 \geq 0,$$

where $m \geq 2$, $(\theta_1, \theta_2) \subseteq (-\infty, \infty)$ and $f\colon (\theta_1,\theta_2) \times (0,\infty) \times (-\infty,\infty) \to (0,\infty)$ satisfies

$$\text{(H)}\qquad \begin{array}{l} f(t,x,y) \text{ is locally Lipschitz continuous for } (x,y) \text{ in}\\ (0,\infty) \times \{(-\infty,0) \cup (0,\infty)\};\ f(t,x,y)/x^{m-1}\\ \text{is strictly decreasing with respect to } x \in (0,\infty) \text{ for}\\ \text{each fixed } (t,y) \in (\theta_1,\theta_2) \times (-\infty,\infty); \text{and}\\ \operatorname{sgn}(y) f(t,x,y) \text{ is decreasing with respect to } y \in (-\infty,\infty)\\ \text{for each fixed } (t,x) \in (\theta_1,\theta_2) \times (0,\infty). \end{array}$$

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Furthermore, we use the uniqueness theorems of (E) with respect to the boundary conditions (BC.i) ($i = 1, 2, 3$) to show that

$$\text{(BVP)} \qquad \begin{cases} \text{(E)}\ (|u'|^{m-2}u')' + f(t,u,u') = 0, \quad \text{in } (\theta_1, \theta_2),\ m \geq 2, \\ \text{(BC)} \begin{cases} \alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) = 0, \\ \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i \geq 0$ and $\alpha_i^2 + \beta_i^2 \neq 0$ ($i = 1, 2$) has at most one positive solution in $C^1([\theta_1, \theta_2])$.

Equations of the type (E) arise in studies of radially symmetric solutions (i.e., solutions u that depend only on the variable $r = |x|$) of the m-Laplace equation,

$$\text{(E}_1\text{)} \quad \nabla \cdot (|\nabla u|^{m-2}\nabla u) + g(|x|, u, \nabla u) = 0, \qquad R_0 < |x| < R_1,\ x \in \mathbb{R}^N,\ N \geq 2.$$

A radially symmetric solution of (E_1) satisfies the ordinary differential equation

$$\text{(E}_2\text{)} \qquad (|u'|^{m-2}u')' + \frac{N-1}{r}|u'|^{m-2}u' + g(r,u,u') = 0, \qquad R_0 < r < R_1.$$

With the change of variables $t = r^{\frac{m-N}{m-1}}$ (for $m \neq N$) or $t = \log r$ (for $m = N$), equation (E_2) can be reduced to an equation of the type (E) or

$$\text{(E}^*\text{)} \qquad (m-1)|u'|^{m-2}u'' + f(t,u,u') = 0, \qquad \theta_1 < t < \theta_2,\ m \geq 2.$$

Conditions for the existence of solutions of equation (E) with respect to (BC.1)–(BC.3) were studied by many authors; see for instance, De Figueiredo, Lions and Nussbaum [6], Granas, Guenther and Lee [9], Kaper, Knaap and Kwong [12], Lions [16], del Pino, Elgueta and Manasevich [19], Rabinowitz [21], Wong [24], and the references therein. The uniqueness problem concerning (E), for the case $m = 2$, has been studied by many authors. For example, Gatica, Oliker and Waltman [7], Kwong [14], Dalmasso [4, 5], Brezis and Oswald [3], Krasnoselskii [13], and the excellent book by Agarwal and Lakshmikantham [1]. However, it seems that very little is known for the case $m \neq 2$. Recently, Naito [17] considered the case $f(t,u,u') = p(t)f(u)$ and established some excellent conditions for uniqueness by using the generalized Prüfer transformation and comparison theorems. In this article, the author attempts to afford a concise approach to study the uniqueness of positive solutions of (E) with boundary conditions (BC.1)–(BC.3) and (BC).

For other related results, we refer the reader to Bobisud [2], Dalmasso [4, 5] Guedda and Veron [10], Naito [17], del Pino and Manasevich [20], O'Regan [22], and Wong and Yu [23].

2. Main result

Let u and v be two distinct positive solutions of (E). We define

$$(1) \qquad w(t) := \{u(t)\}^{m-1}(|v'(t)|^{m-2}v'(t)) - (|u'(t)|^{m-2}u'(t))\{v(t)\}^{m-1}$$

for $t \in [a,b] \subseteq [\theta_1, \theta_2]$.

It is clear that $w(t)$ satisfies

$$\begin{aligned} w'(t) &= u^{m-1}(|v'|^{m-2}v')' - (|u'|^{m-2}u')'v^{m-1} \\ &\quad + (m-1)u'v'\{|v'|^{m-2}u^{m-2} - |u'|^{m-2}v^{m-2}\} \\ &= u^{m-1}\{-f(t,v,v')\} - \{-f(t,u,u')\}v^{m-1} \\ &\quad + (m-1)u'v'\{|v'u|^{m-2} - |u'v|^{m-2}\} \\ &= (uv)^{m-1}\left\{\frac{f(t,u,u')}{u^{m-1}} - \frac{f(t,v,v')}{v^{m-1}}\right\} \\ &\quad + (m-1)u'v'\{|v'u|^{m-2} - |u'v|^{m-2}\} \\ &= (uv)^{m-1}\left\{\frac{f(t,u,u')}{u^{m-1}} - \frac{f(t,v,u')}{v^{m-1}} + \frac{f(t,v,u')}{v^{m-1}} - \frac{f(t,v,v')}{v^{m-1}}\right\} \\ &\quad + (m-1)u'v'\{|v'u|^{m-2} - |u'v|^{m-2}\} \end{aligned} \tag{2}$$

for $t \in (a,b) \subseteq (\theta_1,\theta_2)$.

In order to treat our main results, we need the following:

Lemma 2.1. *Let u and v be distinct positive solutions of* (E)–(BC.i) *in $C^1([\theta_1,\theta_2])$ and $u > v$ on (θ_1,θ_2) for $i = 1,2,3$. Then $w'(t) < 0$ in (θ_1,θ_2), that is, $w(t)$ is strictly decreasing in $[\theta_1,\theta_2]$.*

Proof. We separate the proof into the following cases:

Case (1). Suppose that u and v are two distinct positive solutions of (E)–(BC.1). First, we claim that $u'(t)v(t) \geq v'(t)u(t)$ (≥ 0) in $[\theta_1,\theta_2]$.

(1°) Assume that there exists $t_1 \in (\theta_1,\theta_2)$ such that

$$u'(t)v(t) \geq v'(t)u(t) \ (\geq 0) \text{ on } [\theta_1,t_1) \quad \text{and} \quad u'(t_1)v(t_1) = v'(t_1)u(t_1) \ (\geq 0),$$

which imply $u'(t) \geq v'(t)$ (≥ 0) on $[\theta_1,t_1]$. It follows from (2), $f(t,x,y)/x^{m-1}$ is strictly decreasing with respect to $x \in (0,\infty)$ and $f(\cdot,\cdot,y)$ is decreasing in $(0,\infty)$ that $w'(t) < 0$ on (θ_1,t_1). Thus $w(t)$ is a strictly decreasing function on $[\theta_1,t_1]$. Therefore

$$\begin{aligned} 0 &= (v'(t_1)u(t_1))^{m-1} - (u'(t_1)v(t_1))^{m-1} = w(t_1) \\ &< w(\theta_1) = \{u(\theta_1)\}^{m-1}[|v'(\theta_1)|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] \leq 0, \end{aligned}$$

which gives a contradiction.

(2°) Assume that there exists a strictly decreasing sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} t_n = \theta_1$, $(u'v - v'u)(t_n) = 0$, $(u'v - v'u)'(t_{2n}) = (u''v - v''u)(t_{2n}) \leq 0$ and $(u'v - v'u)'(t_{2n-1}) = (u''v - v''u)(t_{2n-1}) \geq 0$ for all $n \in \mathbb{N}$. It follows from (E*), (H) and $u'(t_{2n}) \geq v'(t_{2n}) \geq 0$ that

$$\begin{aligned} 0 &\geq (m-1)|u'|^{m-2}v^{m-2}(u''v - v''u)(t_{2n}) \\ &= (uv)^{m-1}(t_{2n})\left\{\frac{f(t_{2n},v(t_{2n}),v'(t_{2n}))}{v^{m-1}(t_{2n})} - \frac{f(t_{2n},u(t_{2n}),u'(t_{2n}))}{u^{m-1}(t_{2n})}\right\} \\ &= (uv)^{m-1}(t_{2n})\left\{\frac{f(t_{2n},v(t_{2n}),v'(t_{2n}))}{v^{m-1}(t_{2n})} - \frac{f(t_{2n},u(t_{2n}),v'(t_{2n}))}{u^{m-1}(t_{2n})}\right\} \\ &\quad + (uv)^{m-1}(t_{2n})\left\{\frac{f(t_{2n},u(t_{2n}),v'(t_{2n}))}{u^{m-1}(t_{2n})} - \frac{f(t_{2n},u(t_{2n}),u'(t_{2n}))}{u^{m-1}(t_{2n})}\right\} > 0. \end{aligned}$$

This gives a contradiction. Hence, we have $u'(t)v(t) \geq v'(t)u(t)$ (≥ 0) in $[\theta_1,\theta_2]$, which implies $w'(t) < 0$ in (θ_1,θ_2).

Case (2). Suppose that u and v are two distinct positive solutions of (E)–(BC.2). The proof is quite similar to Case (1), thus we omit the details.

Case (3). Suppose that u and v are two distinct positive solutions of (E)–(BC.3). By virtue of Case (1) and Case (2), we need only consider the case $u'(\theta_1) \neq v'(\theta_1)$ and $u'(\theta_2) \neq v'(\theta_2)$. Without loss of generality, we may assume that $\xi_1 \leq \xi_2$ (resp. $\xi_2 \leq \xi_1$), which implies $u'(\theta_1) > v'(\theta_1) \geq 0$ (resp. $u'(\theta_2) < v'(\theta_2) \leq 0$).

(3°) Assume that there exist $t_1, t_2 \in (\theta_1, \theta_2)$ such that $u'(t_1) = v'(t_2) = 0$. Since $|u'|^{m-2}u'$ and $|v'|^{m-2}v'$ are strictly decreasing in (θ_1, θ_2), t_1 and t_2 are determined uniquely. If $t_1 < t_2$, it follows from $u'(\theta_1) > v'(\theta_1) \geq 0$ and $u'(t_1) = 0 < v'(t_1)$ that there exists $t_3 \in (\theta_1, t_1)$ satisfying

$$u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_3) \quad \text{and} \quad u'(t_3) = v'(t_3) > 0.$$

It follows from $u(\theta_1) = v(\theta_1) = \xi_1 \geq 0$, $u'(t_3) = v'(t_3) > 0$ and Case (1) that $w'(t) < 0$ on (θ_1, t_3). Thus $w(t)$ is a strictly decreasing function on $[\theta_1, t_3]$. Therefore

$$\begin{aligned} 0 &< \{u'(t_3)\}^{m-1}[\{u(t_3)\}^{m-1} - \{v(t_3)\}^{m-1}] = w(t_3) \\ &< w(\theta_1) = \{u(\theta_1)\}^{m-1}[|v'(\theta_1)|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] \leq 0, \end{aligned}$$

which gives a contradiction. If $t_1 > t_2$, it follows from $u'(\theta_2) < v'(\theta_2) \leq 0$ and $v'(t_1) < 0 = u'(t_1)$ that there exists $t_4 \in (t_1, \theta_2)$ such that

$$u'(t_4) = v'(t_4) < 0 \quad \text{and} \quad u'(t) < v'(t) \leq 0 \text{ on } (t_4, \theta_2].$$

It follows from $u(\theta_2) = v(\theta_2) = \xi_2 \geq 0$, $u'(t_4) = v'(t_4) < 0$ and Case (2) that $w'(t) < 0$ on (t_4, θ_2). Thus $w(t)$ is a strictly decreasing function on $[t_4, \theta_2]$. Therefore

$$\begin{aligned} 0 &\leq \{u(\theta_2)\}^{m-1}[|v'(\theta_2)|^{m-2}v'(\theta_2) - |u'(\theta_2)|^{m-2}u'(\theta_2)] = w(\theta_2) \\ &< w(t_4) = |u'(t_4)|^{m-2}u'(t_4)[\{u(t_4)\}^{m-1} - \{v(t_4)\}^{m-1}] \leq 0, \end{aligned}$$

which gives a contradiction, too. Thus, $t_1 = t_2$. By Cases (1)–(2), we see that $w'(t) < 0$ in (θ_1, θ_2).

(4°) Assume that there exists $t_1 \in (\theta_1, \theta_2)$ such that $u'(t_1) = 0$ and $v'(t) \neq 0$ in (θ_1, θ_2). It follows from $u'(\theta_1) > v'(\theta_1)$ (≥ 0) and $u'(t_1) = 0 < v'(t_1)$ that there exists $t_5 \in (\theta_1, t_1)$ satisfying

$$u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_5) \quad \text{and} \quad u'(t_5) = v'(t_5) > 0.$$

Just as in the proof in (3°), we get a contradiction.

(5°) Assume that there exists $t_2 \in (\theta_1, \theta_2)$ such that $v'(t_2) = 0$ and $u'(t) \neq 0$ in (θ_1, θ_2). Therefore, we obtain $0 \leq u'(\theta_2) < v'(\theta_2) \leq 0$, which gives a contradiction.

(6°) Assume that $u'(t) \neq 0$ and $v'(t) \neq 0$ in (θ_1, θ_2). It follows from $u'(\theta_1) > v'(\theta_1)$ (≥ 0) and $u'(\theta_2) < v'(\theta_2)$ (≥ 0) that there exists $t_6 \in (\theta_1, \theta_2)$ satisfying

$$u'(t) > v'(t) > 0 \text{ on } [\theta_1, t_6) \quad \text{and} \quad u'(t_6) = v'(t_6) > 0.$$

Just as in the proof in (3°), we get a contradiction.

Theorem 2.2. *The boundary value problem* (E)–(BC.1) *has at most one positive solution in* $C^1([\theta_1, \theta_2])$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)–(BC.1). We claim that u and v intersect in (θ_1, θ_2). Suppose, on the contrary, that $u(t) > v(t)$ in (θ_1, θ_2). It follows from Lemma 2.1 and $u'(\theta_1) \geq v'(\theta_1)$ (≥ 0) that

$$\begin{aligned} 0 &\leq [\{u(\theta_2)\}^{m-1} - \{v(\theta_2)\}^{m-1}]|v'(\theta_2)|^{m-2}v'(\theta_2) = w(\theta_2) \\ &< w(\theta_1) = \{u(\theta_1)\}^{m-1}[|v'(\theta_1)|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] \leq 0, \end{aligned}$$

which gives a contradiction. Hence, there exists $t_1 \in (\theta_1, \theta_2)$ such that $u(t_1) = v(t_1) > 0$. Following from $u(t_1) = v(t_1) > 0$, $u'(\theta_2) = v'(\theta_2) = \xi_2 \geq 0$, and repeating the same process as above, we obtain a $t_2 \in (t_1, \theta_2)$ such that $u(t_2) = v(t_2) > 0$.

Now, we claim that u and v intersect in (t_1, t_2). Assume, on the contrary, that $u(t) > v(t)$ in (t_1, t_2); then $u'(t_1) \geq v'(t_1) \geq 0$ and $0 \leq u'(t_2) \leq v'(t_2)$. From Lemma 2.1 we see that

$$\begin{aligned} 0 &\leq \{u(t_2)\}^{m-1}[|v'(t_2)|^{m-2}v'(t_2) - |u'(t_2)|^{m-2}u'(t_2)] = w(t_2) \\ &< w(t_1) = \{u(t_1)\}^{m-1}[|v'(t_1)|^{m-2}v'(t_1) - |u'(t_1)|^{m-2}u'(t_1)] \leq 0, \end{aligned}$$

which gives a contradiction, too. Hence, there exists $t_3 \in (t_1, t_2)$ such that $u(t_3) = v(t_3) > 0$. Repeating the same argument, we obtain a strictly decreasing sequence $\{t_n\}_{n=3}^{\infty} \subset (t_1, t_2) \subset (\theta_1, \theta_2)$ such that $t_n \in (t_1, t_{n-1})$ and $u(t_n) = v(t_n)$ for all $n = 3, 4, \dots$. By the Bolzano-Weierstrass theorem, we see that $\{t_n\}_{n=3}^{\infty}$ has an accumulation point, say η, in $[t_1, t_2]$. It is clear that $u(\eta) = v(\eta) > 0$ and $u'(\eta) = v'(\eta) > 0$. Since $f(t, x, y)$ satisfies (H), it follows from the uniqueness of the non-zero initial value problem that $u(t) = v(t)$ in $[\theta_1, \theta_2]$ (see, for example, Hartman [11]).

Theorem 2.3. *The boundary value problem* (E)–(BC.2) *has at most one positive solution in* $C^1([\theta_1, \theta_2])$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)–(BC.2). Just as in the proof of Theorem 2.2, we claim that u and v intersect in (θ_1, θ_2). Suppose, on the contrary, that $u(t) > v(t)$ in (θ_1, θ_2). It follows from Lemma 2.1 and $u'(\theta_2) \leq v'(\theta_2)$ (≤ 0) that

$$\begin{aligned} 0 &\leq \{u(\theta_2)\}^{m-1}[|v'(\theta_2)|^{m-2}v'(\theta_2) - |u'(\theta_2)|^{m-2}u'(\theta_2)] = w(\theta_2) \\ &< w(\theta_1) = [\{u(\theta_1)\}^{m-1} - \{v(\theta_1)\}^{m-1}]|v'(\theta_1)|^{m-2}v'(\theta_1) \leq 0, \end{aligned}$$

which gives a contradiction. Hence, there exists $t_1 \in (\theta_1, \theta_2)$ such that $u(t_1) = v(t_1) > 0$. Since $u(t_1) = v(t_1) > 0$ and $u'(\theta_2) = v'(\theta_2) = \xi_2 \geq 0$, it follows from Theorem 2.2 that $u(t) = v(t)$ on $[t_1, \theta_2]$. Therefore, $u(t) = v(t)$ in $[\theta_1, \theta_2]$.

Theorem 2.4. *The boundary value problem* (E)–(BC.3) *has at most one positive solution in* $C^1([\theta_1, \theta_2])$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)–(BC.3). By virtue of Theorems 2.2 and 2.3, we see that $u'(\theta_1) \neq v'(\theta_1)$ and $u'(\theta_2) \neq v'(\theta_2)$. Without loss of generality, we may assume that $u(t) > v(t)$ in (θ_1, θ_2). Thus $u'(\theta_1) > v'(\theta_1)$. Define t_1 and t_2 so that $u'(t_1) = v'(t_2) = 0$. Similar to the proof of Lemma 2.1, we have that $t_1 = t_2$. Applying Theorems 2.2 and 2.3, we obtain $u \equiv v$ on $[\theta_1, t_1]$ and $u \equiv v$ on $[t_1, \theta_2]$. Therefore, we obtain the desired results.

Theorem 2.5. *The boundary value problem* (BVP) *has at most one positive solution in* $C^1([\theta_1, \theta_2])$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (BVP). We split the proof into the following cases.

Case (1). Assume that $\alpha_1 = 0$, that is, $u'(\theta_1) = v'(\theta_1) = 0$. Since $|u'|^{m-2}u'$ and $|v'|^{m-2}v'$ are strictly decreasing in (θ_1, θ_2), $u'(t) < 0$ and $v'(t) < 0$ in $(\theta_1, \theta_2]$. Now, we claim that u and v intersect in (θ_1, θ_2). Suppose to the contrary that $u(t) > v(t) > 0$ in (θ_1, θ_2).

(1°) If $\alpha_2 = 0$, then $u'(\theta_2) = v'(\theta_2) = 0$. This contradicts the fact that $u'(t) < 0$ and $v'(t) < 0$ in $(\theta_1, \theta_2]$.

(2°) If $\beta_2 = 0$, then $u(\theta_2) = v(\theta_2) = 0$. It follows from $u'(\theta_1) = v'(\theta_1) = 0$, $u(\theta_2) = v(\theta_2) = 0$ and Theorem 2.3 that $u(t) = v(t)$ on $[\theta_1, \theta_2]$, which gives a contradiction.

(3°) If $\alpha_2\beta_2 \neq 0$, then $u'(\theta_2)v(\theta_2) = v'(\theta_2)u(\theta_2)$. It is clear that $u(\theta_2) > v(\theta_2) > 0$, and thus $u'(\theta_2) < v'(\theta_2) < 0$. In fact, if $v(\theta_2) = 0$ (resp. $u(\theta_2) = v(\theta_2)$), it follows from (BC) and $u'(\theta_2)v(\theta_2) = v'(\theta_2)u(\theta_2)$ that $u(\theta_2) = v(\theta_2) \geq 0$, with Theorem 2.3 gives a contradiction.

Repeating the similar argument in Case (1) of Lemma 2.1, we can see that $w(t)$ is strictly decreasing on $[\theta_1, \theta_2]$. Therefore, we obtain

$$\begin{aligned}
0 &= (-1)^{m-2}[(v'(\theta_1)u(\theta_1))^{m-1} - (u'(\theta_1)v(\theta_1))^{m-1}] = w(\theta_1) \\
&> w(\theta_2) = (-1)^{m-2}[(v'(\theta_2)u(\theta_2))^{m-1} - (u'(\theta_2)v(\theta_2))^{m-1}] = 0,
\end{aligned}$$

which gives a contradiction.

Hence, there is $t_2 \in (\theta_1, \theta_2)$ such that $u(t_2) = v(t_2) > 0$. Since $u'(\theta_1) = v'(\theta_1) = 0$ and $u(t_2) = v(t_2) > 0$, it follows from Theorem 2.3 that $u(t) = v(t)$ on $[\theta_1, t_2]$. Therefore, $u(t) = v(t)$ in $[\theta_1, \theta_2]$.

Case (2). Assume that $\beta_1 = 0$, that is, $u(\theta_1) = v(\theta_1) = 0$. It follows from $|u'|^{m-2}u'$, $|v'|^{m-2}v'$ strictly decreasing in (θ_1, θ_2) and $u(\theta_1) = v(\theta_1) = 0$ that $u'(\theta_1) > 0$ and $v'(\theta_1) > 0$. Now, we claim that u and v intersect in (θ_1, θ_2). Suppose to the contrary that $u(t) > v(t) > 0$ in (θ_1, θ_2), and this implies $u'(\theta_1) \geq v'(\theta_1) > 0$.

(4°) If $\alpha_2 = 0$, then $u'(\theta_2) = v'(\theta_2) = 0$. It follows from Theorem 2.2 that $u(t) = v(t)$ in $[\theta_1, \theta_2]$, which gives a contradiction.

(5°) If $\beta_2 = 0$, then $u(\theta_2) = v(\theta_2) = 0$. It follows from Theorem 2.4 that $u(t) = v(t)$ in $[\theta_1, \theta_2]$, which gives a contradiction.

(6°) If $\alpha_2\beta_2 \neq 0$, then $u'(\theta_2)v(\theta_2) = v'(\theta_2)u(\theta_2)$. Repeating the same argument in Case (1) of Lemma 2.1 (or cf. Case (1)–(3°)), we see that $w(t)$ is strictly decreasing on $[\theta_1, \theta_2]$. Therefore, we obtain

$$\begin{aligned}
0 &= \{u(\theta_1)\}^{m-1}[|(v'(\theta_1)|^{m-2}v'(\theta_1) - |u'(\theta_1)|^{m-2}u'(\theta_1)] = w(\theta_1) \\
&> w(\theta_2) = \{u(\theta_2)\}^{m-1}(|v'(\theta_2)|^{m-2}v'(\theta_2)) - (|u'(\theta_2)|^{m-2}u'(\theta_2))\{v(\theta_2)\}^{m-1} = 0,
\end{aligned}$$

which gives a contradiction.

Hence, there is $t_2 \in (\theta_1, \theta_2)$ such that $u(t_2) = v(t_2) > 0$. Since $u(\theta_1) = v(\theta_1) = 0$ and $u(t_2) = v(t_2) > 0$, it follows from Theorem 2.4 that $u(t) = v(t)$ on $[\theta_1, t_2]$. Therefore, $u(t) = v(t)$ in $[\theta_1, \theta_2]$.

Just as in the proof of Cases (1)–(2), we can exclude the possibility of $\alpha_2 = 0$ or $\beta_2 = 0$.

Case (3). Assume that $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$, that is, $u'(\theta_i)v(\theta_i) = v'(\theta_i)u(\theta_i)$ for $i = 1, 2$. The rest of the proof is quite similar to the proofs in Cases (1)–(3°) and Cases (2)–(6°), so we omit the details.

By Cases (1)–(3), we obtain the desired results.

3. Remarks and examples

Recently, Gatica, Oliker and Waltman [7], Kwong [14], Naito [17], Brezis and Oswald [3], and Dalmasso [5] showed the following important results:

Theorem 3.A ([3, Theorem 1]). *Consider the problem*

$$\text{(BVP.1)}\qquad \begin{cases} \Delta u + f(x,u) = 0 \quad \textit{in } \Omega, \\ u \geq 0,\ u \not\equiv 0 \quad \textit{in } \Omega, \\ u = 0 \quad \textit{on } \partial\Omega, \end{cases}$$

and make the following assumptions:

(A_1) *For a.e.* $x \in \Omega$ *the function* $u \to f(x,u)$ *is continuous on* $[0,\infty)$ *and the function* $u \to f(x,u)/u$ *is strictly decreasing in* $(0,\infty)$.

(A_2) *For each* $u \geq 0$ *the function* $u \to f(x,u)$ *belongs to* $L^\infty(\Omega)$. *Then, there exists at most one solution of* (BVP.1) *in* $H_0^1 \cap L^\infty(\Omega)$.

Theorem 3.B ([5, Theorem 1]). *Let* $f \in C^1([0,R] \times [0,\infty))$ *satisfying*

(C_1) $uf_u(t,u) > f(t,u)$ *for* $(t,u) \in [0,R] \times [0,\infty)$, *that is,* $\frac{f(t,u)}{u}$ *is strictly increasing in* $u \in [0,\infty)$ *for each fixed* $t \in [0,R]$;

(C_2) $f_t(u,u) \leq 0$ *for* $(t,u) \in [0,R] \times [0,\infty)$;

(C_3) *there exists* $u > 0$ *such that* $f(R,u) \geq 0$.

Then

$$\text{(BVP.2)}\qquad \begin{cases} u''(t)) + f(|t|, u(t)) = 0, \qquad -R < t < R, \\ u(-R) = u(R) = 0 \end{cases}$$

has at most one positive solution in $C^2[-R,R]$.

Theorem 3.C ([7, Theorem 4.1]). *Let* $k \in \{1,2,3,\dots\}$, $p \in (0,1)$ *and* $h \in C([0,1);\ [0,\infty))$ *such that*

$$0 < \int_0^1 (1-t)^{-p} h(t)dt < \infty.$$

Then

$$\text{(E}_3\text{)}\qquad u''(t) + \frac{k}{t}u'(t) + h(t)u^{-p}(t) = 0 \quad \textit{in } (0,1)$$

has at least one positive solution satisfying (BC.2) *in* $C^1[0,1] \cap C^2(0,1)$.

Theorem 3.D ([14, Theorem 2]). *Assume that* $q(t) > 0$ *in* $(0,1)$ *and* $f(u)/u$ *is decreasing in* $(0,\infty)$ *and not constant in any neighborhood of* $u = 0$. *Then*

$$\text{(E}_4\text{)}\qquad u''(t) + q(t)f(u(t)) = 0 \quad \textit{in } (0,1)$$

has a unique positive solution satisfying (BC.2) *in* $C^1[0,1] \cap C^2(0,1)$.

Theorem 3.E ([17, Theorems 1–3]). *Let*

(B_1) $p \in C[\theta_1,\theta_2]$ *and* $p(t) > 0$ *in* (θ_1,θ_2);

(B_2) $f \in C[0,\infty)$, $f(u) > 0$ *in* $(0,\infty)$ *and* $\frac{f(u)}{u^{m-1}}$ *is decreasing in* $u \in (0,\infty)$;

(B_3) *for any* $\lambda > 0$ (*resp.* $\lambda < 0$), *a solution* u *of*

$$\text{(E}_5\text{)}\qquad (|u'|^{m-2}u')' + p(t)f(u) = 0, \qquad \theta_1 < t < \theta_2,\ m \geq 2,$$

satisfying $u(\theta_1) = 0$ *and* $u'(\theta_1) = \lambda$ *(resp.* $u(\theta_2) = 0$ *and* $u'(\theta_2) = \lambda$*) is determined uniquely as long as* $u'(t) > 0$ *(resp.* $u'(t) < 0$*).*

Then, (E_5) *has at most one positive solution in* $C^1[\theta_1, \theta_2]$ *satisfying* (BC.1)–(BC.3).

Remark 3.F. Comparing our uniqueness theorems with the above-mentioned results, we have the following remarks:

(I) The assumption "$u \to f(x, u)$ is continuous on $[0, \infty)$" in Theorem 3.A implies $f(x, u) \neq u^p$ for $p < 0$, and "$f(u)/u$ is decreasing in $(0, \infty)$ and is not a constant in any neighborhood of $u = 0$" in Theorem 3.D is imposed to exclude the situation in which $f(u)$ behaves like a linear function in a neighborhood of $u = 0$, that is, $f(u)/u$ behaves like a strictly decreasing function near $u = 0$.

(II) It is clear that if $f(t, u)$ is (strictly) decreasing in $u \in (0, \infty)$ and

$$f(t, u) \equiv h(t)u^{-p}, h(t)u^q, u^\alpha + u^{-\alpha}, \sin(t)u^{-p} + \cos(t)u^q$$

for any given $p \in [0, \infty)$, $q \in [0, m - 1)$, $\alpha \in [0, m - 1]$ and $h \in C((0, 1); [0, \infty))$, then $f(t, u)$ satisfies "$f(t, u)/u^{m-1}$ is strictly decreasing in u". But Theorems 3.A, 3.C, 3.D and 3.E cannot be applied to most of these functions, for example,

$$f(t, u) \equiv u^{1/2}, u^{-1} + u \text{ and } tu^{\frac{1}{3}} + e^t u^{\frac{1}{2}}.$$

Furthermore, Theorem 3.C does not tell us "the uniqueness of positive solution of (E_3) with (BC.2)".

(III) Our main results generalize Theorems 3.A, 3.C, 3.D, 3.E and also confirm the uniqueness of Theorem 3.C.

(IV) The excellent uniqueness Theorem 3.B of Dalmasso [5] combines with the main results in this article. They can criticize almost all the uniqueness of positive solutions of (E)–(BC.i) ($i = 1, 2, 3$) and (BVP).

Example 3.G. (I) It follows from Theorem 2.2 (resp. Theorems 2.3, 2.4, 2.5) that the boundary value problem

(BVP.3)
$$\begin{cases} (|u'|^{m-2}u')' + 2[t(1-t)]^2 u^{-p} + \frac{\sin(t)|u'|^{\frac{-1}{3}}}{1+t^2} = 0 \quad \text{in } (0, 1), \\ u(0) = u'(1) = 0 \\ (\text{resp. } u'(0) = u(1) = 0,\ u(0) = u(1) = 0,\ u(0) = u(1) + 2u'(1) = 0) \end{cases}$$

has at most one positive solution in $C^1[0, 1]$, where $p \in (0, \infty)$.

(II) It follows from Theorem 2.3 (resp. Theorems 2.2, 2.4, 2.5) that the boundary value problem

(BVP.4)
$$\begin{cases} (|u'|^{m-2}u')' + \frac{1}{1+t^2} u^p = 0 \quad \text{in } (0, 1),\ p \in (-\infty, m - 1), \\ u'(0) = u(1) = 0 \\ (\text{resp. } u(0) = u'(1) = 0,\ u(0) = u(1) = 0,\ u(0) = u(1) + 2u'(1) = 0) \end{cases}$$

has at most one positive solution in $C^1[0, 1]$.

(III) It follows from Theorem 2.4 (resp. Theorems 2.2, 2.3, 2.5) that the boundary value problem

$$\text{(BVP.5)}\quad \begin{cases} (|u'|^{m-2}u')' + \frac{1}{t^{\alpha+1}}(u^{\alpha} + u^{-\alpha}) + \cos(t)|u'|^{-\beta} = 0 \quad \text{in } (0,1), \\ u(0) = u(1) = 0 \\ (\text{resp. } u(0) = u'(1) = 0, u'(0) = u(1) = 0, u(0) = u(1) + 2u'(1) = 0) \end{cases}$$

has at most one positive solution in $C^1[0,1]$, where $\alpha \in [0, m-1]$, $\beta > 0$.

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