

## ON SELF-ADJOINTNESS OF A SCHRÖDINGER OPERATOR ON DIFFERENTIAL FORMS

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ABSTRACT. Let  $M$  be a complete Riemannian manifold and let  $\Omega^\bullet(M)$  denote the space of differential forms on  $M$ . Let  $d : \Omega^\bullet(M) \rightarrow \Omega^{\bullet+1}(M)$  be the exterior differential operator and let  $\Delta = dd^* + d^*d$  be the Laplacian. We establish a sufficient condition for the Schrödinger operator  $H = \Delta + V(x)$  (where the potential  $V(x) : \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$  is a zero order differential operator) to be self-adjoint. Our result generalizes a theorem by I. Oleinik about self-adjointness of a Schrödinger operator which acts on the space of scalar valued functions.

### 1. INTRODUCTION

Suppose  $M$  is a complete Riemannian non-compact manifold. We will assume that  $M$  is oriented and connected. Let  $T^*M$  denote the cotangent bundle to  $M$  and let  $\bigwedge^\bullet(T^*M) = \bigoplus_i \bigwedge^i(T^*M)$  denote the exterior algebra of  $T^*M$ . We denote by  $L^2\Omega^\bullet(M)$  the space of square integrable complex valued differential forms on  $M$ , i.e. the space of sections of  $\bigwedge^\bullet(T^*M) \otimes \mathbb{C}$  which are square integrable with respect to the scalar product

$$(1) \quad \langle \alpha, \beta \rangle = \int_M \alpha \wedge * \bar{\beta}, \quad \alpha, \beta \in L^2\Omega^\bullet(M).$$

Here  $*$  denotes the Hodge operator associated to the Riemannian metric on  $M$ . Note that  $L^2\Omega^0(M)$  is just the space of square integrable complex valued functions on  $M$ .

Let  $d : L^2\Omega^\bullet(M) \rightarrow L^2\Omega^{\bullet+1}(M)$  denote the exterior differential and let  $d^*$  be the operator formally adjoint to  $d$  with respect to the scalar product (1).

Let  $\Delta = dd^* + d^*d$  be the Laplacian and consider the Schrödinger operator

$$(2) \quad H = \Delta + V(x) : L^2\Omega^\bullet(M) \rightarrow L^2\Omega^\bullet(M)$$

where the *potential*  $V(x)$  is a measurable section of the bundle  $\text{End}(\bigwedge^\bullet(T^*M))$  of endomorphisms of  $\bigwedge^\bullet(T^*M)$  which belongs to the class  $L^\infty_{loc}$  (i.e. such that for any compact set  $K \subset M$  there exists a constant  $C_K > 0$  such that  $|V(x)| \leq C_K$  for almost all  $x \in K$ ).

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We denote by  $H_0$  the restriction of  $H$  on the space  $\Omega_c^\bullet(M)$  of smooth differential forms with compact support. The purpose of this paper is to introduce a sufficient condition on the potential  $V(x)$  for operator  $H_0$  to be self-adjoint.

## 2. STATEMENT OF RESULTS

For  $x, y \in M$  let  $\text{dist}(x, y)$  denote the Riemannian distance between  $x$  and  $y$ . Fix a point  $p \in M$  and set  $r(x) = \text{dist}(x, p)$ .

Fix  $x \in M$ . The Riemannian metric on  $M$  defines a scalar product  $\langle \cdot, \cdot \rangle_x$  on the fiber  $\bigwedge^\bullet(T_x^*M) \otimes \mathbb{C}$  of the vector bundle  $\bigwedge^\bullet(T^*M) \otimes \mathbb{C}$ . As usual, we write  $V(x) \geq C$  if

$$(3) \quad \langle V(x)\xi, \xi \rangle_x \geq C \langle \xi, \xi \rangle_x$$

for any  $\xi \in \bigwedge^\bullet(T_x^*M) \otimes \mathbb{C}$ . Note that it follows from (3) that  $V(x)$  is a self-adjoint endomorphism of  $\bigwedge^\bullet(T_x^*M)$ .

**Theorem A.** *Assume that for almost all  $x \in M$  the potential  $V(x)$  of the operator (2) satisfies the estimate*

$$(4) \quad V(x) \geq -Q(x),$$

where  $1 \leq Q(x) \leq \infty$  and  $Q^{-1/2}(x)$  is a Lipschitz function on  $M$  such that

$$(5) \quad |Q^{-1/2}(x) - Q^{-1/2}(y)| \leq K \text{dist}(x, y) \quad \text{for any } x, y \in M.$$

If for any piecewise smooth curve  $\gamma : [0, \infty) \rightarrow M$  such that  $\lim_{t \rightarrow \infty} r(\gamma(t)) = \infty$  the integral

$$(6) \quad \int_\gamma Q^{-1/2}(x) d\gamma = \infty$$

then the operator  $H_0$  is essentially self-adjoint.

For the case of a Schrödinger operator acting on scalar valued functions this theorem was established by I. Oleinik [O2]. Note that  $Q(x)$  may be equal to infinity on a set of positive measure.

As a simple consequence of Theorem A we obtain the following

**Theorem B.** *Suppose that for almost all  $x \in M$  the potential  $V(x)$  satisfies the estimate  $V(x) \geq -q(r(x))$ , where  $1 \leq q \leq \infty$  and  $q^{-1/2}(t)$  is a Lipschitz function on  $\mathbb{R}$  such that  $\int_0^\infty q^{-1/2}(t) dt = \infty$ . Then the operator  $H_0$  is essentially self-adjoint.*

*In particular, if  $M = \mathbb{R}^n$  and  $V(x) \geq -C|x|^2$  then the operator  $H_0$  is essentially self-adjoint.*

*Remark.* Theorem A remains true if we replace  $L^2\Omega^\bullet(M)$  by the space of square integrable forms on  $M$  with values in a flat Hermitian vector bundle  $\mathcal{F}$  over  $M$ , provided that the Hermitian structure on  $\mathcal{F}$  is flat. In this case the differential  $d$  should be replaced by the covariant differential associated to the flat structure on  $\mathcal{F}$ . The proof is a verbatim repetition of the proof for the scalar case, cf. below. However the notation in the vector valued case is more complicated.

3. HISTORICAL REMARKS

An analogue of Theorem B for the case  $M = \mathbb{R}^1$  was established by Sears [Se]. B. Levitan [Le] proved the Sears theorem for the Schrödinger operator acting on scalar valued functions on  $M = \mathbb{R}^n$ . F. Rofo-Beketov [RB] extended these results to the case where the potential  $V(x)$  cannot be estimated by a function depending only on  $\text{dist}(x, p)$ . Many results and references about the essential self-adjointness of Schrödinger operators on  $\mathbb{R}^n$  may be found in [RS].

I. Oleinik [O1], [O2] established Theorem A for the Schrödinger operator acting on scalar valued functions on a complete Riemannian manifold.

Essential self-adjointness of a pure Laplacian (without lower order terms) on differential forms on a complete Riemannian manifold was first stated and proved by M. P. Gaffney [Ga1], [Ga2]. A much more general statement was proven by P. Chernoff [Ch]. A number of related results may be found in [Sh].

4. THE DOMAIN OF  $\mathcal{D}(H_0^*)$

Let  $H_0^*$  denote the operator adjoint to  $H_0$ . The domain  $\mathcal{D}(H_0^*)$  of  $H_0^*$  consists of forms  $\alpha \in L^2\Omega^\bullet(M)$  such that  $H\alpha$  understood in the sense of distributions also belongs to  $L^2\Omega^\bullet(M)$ .

The operator  $H_0$  is symmetric. Hence, to show that its closure is self-adjoint it is enough to show that the adjoint operator  $H_0^*$  is symmetric. In other words we have to prove that

$$(7) \quad \int_M (H\alpha \wedge *\bar{\beta} - \alpha \wedge *\overline{H\beta}) = 0 \quad \text{for any } \alpha, \beta \in \mathcal{D}(H_0^*).$$

To prove (7) we need some information about the behavior of differential forms from  $\mathcal{D}(H_0^*)$ . The main result of this section is the following lemma, which provides us with this information.

**Lemma 1.** *If  $\alpha \in \mathcal{D}(H_0^*)$  then the forms  $Q^{-1/2}d\alpha, Q^{-1/2}d^*\alpha$  are square integrable.*

*Remark.* 1. By the standard theory of elliptic operators any  $\alpha \in \mathcal{D}(H_0^*)$  belongs to the Sobolev space  $H_{loc}^2$ . Hence,  $d\alpha, d^*\alpha$  are locally square integrable. Thus the lemma provides us with information about the behavior of the forms from  $\mathcal{D}(H_0^*)$  at infinity.

2. For the Schrödinger operator on scalar valued functions on  $\mathbb{R}^n$  an analogous lemma was established in [RB]. The proof was adopted in [O1], [O2] to the case of a Riemannian manifold. In our proof we follow rather closely the lines of [O2]. However, the fact that we deal with differential forms rather than with scalar valued functions demands a more careful analysis.

*Proof.* Recall that we fixed a point  $p \in M$  and that for any  $x \in M$  we denoted by  $r(x)$  the Riemannian distance between  $x$  and  $p$ .

It is shown in [O2, Proof of Lemma 1] that for any  $R > 0, \varepsilon > 0$  there exist smooth functions  $r_{R,\varepsilon}(x), F_{R,\varepsilon}(x)$  on  $M$  which approximate the Lipschitz functions  $r(x), Q^{-1/2}(x)$  in the sense that

$$(8) \quad \begin{aligned} |r_{R,\varepsilon}(x) - r(x)| &< \varepsilon, & Q^{-1/2}(x) - \varepsilon &< F_{R,\varepsilon}(x) < (1 + \varepsilon)Q^{-1/2}(x), \\ \overline{\lim}_{\varepsilon \rightarrow 0} |dr_{R,\varepsilon}(x)| &\leq 1, & \overline{\lim}_{\varepsilon \rightarrow 0} |dF_{R,\varepsilon}(x)| &\leq K, \end{aligned}$$

for any  $x \in r_{R,\varepsilon}^{-1}([0, R + 1])$ . Here  $K$  is the same constant as in (5).

Let  $\Psi : [0, +\infty) \rightarrow [0, 1]$  be a smooth function which is equal to one when  $t \leq 1/2$  and which is equal to zero when  $t \geq 1$ . Set

$$(9) \quad \psi_{R,\varepsilon}(x) = \begin{cases} \Psi\left(\frac{r_{R,\varepsilon}(x)}{R}\right) F_{R,\varepsilon}(x) & \text{if } r_{R,\varepsilon}(x) \leq R; \\ 0 & \text{outside of the set } r_{R,\varepsilon}(x) \leq R. \end{cases}$$

For any  $R > 0$  the functions  $\psi_{R,\varepsilon}$ ,  $\varepsilon < 1$  vanish outside of the compact set  $r^{-1}([0, R+1])$ . Hence, it follows from (8) and (4) that there exist a constant  $K_1 > 0$  not depending on  $R$  and a number  $\varepsilon_R > 0$  (which does depend on  $R$ ) such that

$$(10) \quad |d\psi_{R,\varepsilon}(x)| \leq K_1, \quad \psi_{R,\varepsilon}^2(x) \leq 2, \quad \left| \int_M \psi_{R,\varepsilon}^2 \alpha \wedge *V\alpha \right| \leq 2\|\alpha\|^2,$$

for any  $x \in M$ ,  $R > 1$ ,  $0 < \varepsilon < \varepsilon_R$ ,  $\alpha \in L^2\Omega^\bullet(M)$ . Here  $\|\alpha\| = \langle \alpha, \alpha \rangle^{\frac{1}{2}}$  denotes the  $L^2$ -norm of the form  $\alpha$ .

Functions  $\psi_{R,\varepsilon}$  have compact support. Hence, in view of Remark 1 after the statement of the lemma, the forms  $\psi_{R,\varepsilon}d\alpha$  and  $\psi_{R,\varepsilon}d^*\alpha$  are square integrable. Assume that  $\alpha \in \mathcal{D}(H_0^*)$  is a real valued form and set

$$(11) \quad J_{R,\varepsilon}^2 = \|\psi_{R,\varepsilon}d\alpha\|^2 + \|\psi_{R,\varepsilon}d^*\alpha\|^2 = \int_M \psi_{R,\varepsilon}(x)^2 (d\alpha \wedge *d\alpha + d^*\alpha \wedge *d^*\alpha).$$

It follows from (8), (9) that to prove the lemma it is enough to show that

$$(12) \quad \overline{\lim}_{R \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} J_{R,\varepsilon} < \infty.$$

Let us first rewrite the integrand in (11) in a more convenient form. In the calculations bellow we use the equality (cf. [Wa, §6.1])  $d^*\alpha = (-1)^{|\alpha|} *^{-1} d * \alpha$  where  $|\alpha|$  denotes the degree of the differential form  $\alpha$ .

$$(13) \quad \begin{aligned} \psi_{R,\varepsilon}^2 d\alpha \wedge *d\alpha &= d(\psi_{R,\varepsilon}^2 \alpha \wedge *d\alpha) - 2\psi_{R,\varepsilon} d\psi_{R,\varepsilon} \wedge \alpha \wedge *d\alpha + \psi_{R,\varepsilon}^2 \alpha \wedge *d^*d\alpha, \\ \psi_{R,\varepsilon}^2 d^*\alpha \wedge *d^*\alpha &= (-1)^{|\alpha|} \psi_{R,\varepsilon}^2 d^*\alpha \wedge d * \alpha \\ &= -d(\psi_{R,\varepsilon}^2 d^*\alpha \wedge * \alpha) + 2\psi_{R,\varepsilon} d\psi_{R,\varepsilon} \wedge d^*\alpha \wedge * \alpha + \psi_{R,\varepsilon}^2 dd^*\alpha \wedge * \alpha. \end{aligned}$$

It follows now from (13), (10) and from the Stokes theorem that, if  $R > 1$ ,  $\varepsilon < \varepsilon_R$ , then

$$\begin{aligned} \|\psi_{R,\varepsilon}d\alpha\|^2 &= \int_M \psi_{R,\varepsilon}^2 d\alpha \wedge *d\alpha = -2\langle d\psi_{R,\varepsilon} \wedge \alpha, \psi_{R,\varepsilon}d\alpha \rangle + \langle \alpha, \psi_{R,\varepsilon}^2 d^*d\alpha \rangle \\ &\leq 2K_1\|\alpha\| \|\psi_{R,\varepsilon}d\alpha\| + \langle \alpha, \psi_{R,\varepsilon}^2 d^*d\alpha \rangle, \\ \|\psi_{R,\varepsilon}d^*\alpha\|^2 &= \int_M \psi_{R,\varepsilon}^2 d^*\alpha \wedge *d^*\alpha = 2\langle d\psi_{R,\varepsilon} \wedge \psi_{R,\varepsilon}d^*\alpha, \alpha \rangle + \langle \psi_{R,\varepsilon}^2 dd^*\alpha, \alpha \rangle \\ &\leq 2K_1\|\psi_{R,\varepsilon}d^*\alpha\| \|\alpha\| + \langle \alpha, \psi_{R,\varepsilon}^2 dd^*\alpha \rangle. \end{aligned}$$

Summing these two equations we obtain

$$(14) \quad \begin{aligned} J_{R,\varepsilon}^2 &\leq 2K_1\|\alpha\| (\|\psi_{R,\varepsilon}d\alpha\| + \|\psi_{R,\varepsilon}d^*\alpha\|) + \langle \alpha, \psi_{R,\varepsilon}^2 \Delta\alpha \rangle \\ &\leq 4K_1\|\alpha\| J_{R,\varepsilon} + \int_M \psi_{R,\varepsilon}^2 (\alpha \wedge *H\alpha - \alpha \wedge *V\alpha) \\ &\leq 4K_1\|\alpha\| J_{R,\varepsilon} + 2\|\alpha\| \|H\alpha\| + 2\|\alpha\|^2. \end{aligned}$$

Here the last inequality follows from (10).

It follows from (14) that the set  $\{J_{R,\varepsilon} : R > 1, \varepsilon < \varepsilon_R\}$  is bounded from above. Hence (12) holds. The proof of the lemma is completed.  $\square$

### 5. PROOF OF THEOREM A

We apply a modification of the method used in [RB] suggested by I. Oleinik [O2]. The quantity

$$(15) \quad \tilde{\rho}(x, y) = \inf_{\gamma} \int_{\gamma} Q^{-1/2}(x) d\gamma,$$

where the infimum is taken over all piecewise smooth curves connecting the points  $x, y \in M$ , is called *generalized distance* between  $x$  and  $y$ . It is a symmetric function in  $x, y$  which satisfies the triangular inequality. The first metric axiom is not valid in general. Note, however, that (6) implies that the sets  $P^{-1}([0, R])$  are compact for any  $R > 0$ .

Recall that in Section 2 we have fixed a point  $p \in M$ . Set  $P(x) = \tilde{\rho}(x, p)$ . Then (cf. [O2, Lemma 2])

$$(16) \quad |P(x) - P(y)| \leq Q^{-1/2}(x) \text{dist}(x, y) + \frac{K}{2}(\text{dist}(x, y))^2$$

for any  $x, y \in M$ . It follows (cf. [O2]) that for any  $R > 0, \varepsilon > 0$  there exists a smooth function  $\tilde{P}_{R,\varepsilon}(x)$  which approximates  $P(x)$  in the sense that

$$(17) \quad |\tilde{P}_{R,\varepsilon}(x) - P(x)| \leq \varepsilon, \quad \overline{\lim}_{\varepsilon \rightarrow 0} |d\tilde{P}_{R,\varepsilon}(x)| \leq Q^{-1/2}(x),$$

for any  $x \in P^{-1}([0, R + 1])$ .

Assume that  $\varepsilon < 1$  so that  $\tilde{P}_{R,\varepsilon}^{-1}([0, R]) \subset P^{-1}([0, R + 1])$ . Let us define a piecewise smooth function  $P_{R,\varepsilon}(x)$  on  $M$  by the formula

$$(18) \quad P_{R,\varepsilon}(x) = \begin{cases} \tilde{P}_{R,\varepsilon}(x) & \text{if } \tilde{P}_{R,\varepsilon}(x) \leq R; \\ R & \text{outside the set } \tilde{P}_{R,\varepsilon}(x) \leq R. \end{cases}$$

By (17), the inequality

$$(19) \quad \overline{\lim}_{\varepsilon \rightarrow 0} |dP_{R,\varepsilon}(x)| \leq Q^{-1/2}(x)$$

holds almost everywhere on  $M$ .

Recall from Section 4 that the statement of Theorem A is equivalent to equality (7). Fix  $\alpha, \beta \in \mathcal{D}(H_0^*)$  and consider the following approximation of the integral (7):

$$(20) \quad \begin{aligned} I_{R,\varepsilon} &= \int_M \left(1 - \frac{P_{R,\varepsilon}}{R}\right) (H\alpha \wedge *\bar{\beta} - \alpha \wedge *\overline{H\beta}) \\ &= \int_M \left(1 - \frac{P_{R,\varepsilon}}{R}\right) (\Delta\alpha \wedge *\bar{\beta} - \alpha \wedge *\overline{\Delta\beta}). \end{aligned}$$

By the Fatou theorem ([RS, Theorem I.17]), it is enough to show that

$$(21) \quad \overline{\lim}_{R \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} I_{R,\varepsilon} = 0.$$

We will need the following “integration by parts” lemma.<sup>1</sup>

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<sup>1</sup>I learned this lemma from M. Shubin.

**Lemma 2.** *Let  $\phi : M \rightarrow \mathbb{R}$  be a smooth function with compact support. Then*

$$(22) \quad \begin{aligned} & \int_M \phi \Delta \alpha \wedge * \bar{\beta} \\ &= \int_M \phi (d\alpha \wedge * d\bar{\beta} + d^* \alpha \wedge * d^* \bar{\beta}) + \int_M d\phi \wedge (\bar{\beta} \wedge * d\alpha - d^* \alpha \wedge * \bar{\beta}) \end{aligned}$$

for any  $\alpha, \beta \in \mathcal{D}(H_0^*)$ .

Note that, by Remark 1 after the statement of Lemma 1, all the integrals in (22) make sense.

*Proof.* Recall that  $d^* u = (-1)^{|u|} *^{-1} d * u$  where  $|u|$  denotes the degree of the differential form  $u$ . Hence, if  $|u| = |v| - 1$ , then

$$(23) \quad \phi du \wedge * v = \phi u \wedge * d^* v - d\phi \wedge u \wedge * v + d(\phi u \wedge * v).$$

Substituting into (23) first  $u = d^* \alpha, v = \bar{\beta}$  and then  $u = \bar{\beta}, v = d\alpha$  we obtain

$$\begin{aligned} \phi dd^* \alpha \wedge * \bar{\beta} &= -d\phi \wedge d^* \alpha \wedge * \bar{\beta} + \phi d^* \alpha \wedge * d^* \bar{\beta} + d(\phi d^* \alpha \wedge * \bar{\beta}), \\ \phi d^* d\alpha \wedge * \bar{\beta} &= \phi \bar{\beta} \wedge * d^* d\alpha = d\phi \wedge \bar{\beta} \wedge d\alpha + \phi d\alpha \wedge * d\bar{\beta} - d(\phi \bar{\beta} \wedge * d\alpha). \end{aligned}$$

In the last equality we used that  $u \wedge * v = v \wedge * u$  for any differential forms  $u, v$  of the same degree. Summing the above equations, integrating over  $M$  and using the Stokes theorem we get (22).  $\square$

Using definition (20) of  $I_{R,\varepsilon}$  and Lemma 2 we obtain

$$(24) \quad I_{R,\varepsilon} = \frac{1}{R} \int_M dP_{R,\varepsilon} \wedge (\bar{\beta} \wedge * d\alpha - d^* \alpha \wedge * \bar{\beta} - \alpha \wedge * d\bar{\beta} + d^* \bar{\beta} \wedge * \alpha).$$

Let  $d\mu(x)$  denote the Riemannian density on  $M$ . For any  $\xi \in \bigwedge^k(T^*M) \otimes \mathbb{C}$  we denote by  $|\xi|$  its norm with respect to the scalar product on  $\bigwedge^k(T^*M) \otimes \mathbb{C}$  induced by the Riemannian structure on  $M$ . Then

$$(25) \quad |\langle \alpha, \beta \rangle| \leq \int_M |\alpha \wedge * \bar{\beta}| d\mu(x) \leq \int_M |\alpha| |\beta| d\mu(x) \leq \|\alpha\| \|\beta\|$$

for any  $\alpha, \beta \in L^2 \Omega^\bullet(M)$ .

Let us estimate the behavior of the right-hand side of (24) as  $\varepsilon \rightarrow 0$ . For the first term we obtain

$$(26) \quad \begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \left| \frac{1}{R} \int_M dP_{R,\varepsilon} \wedge \bar{\beta} \wedge * d\alpha \right| &\leq \frac{1}{R} \overline{\lim}_{\varepsilon \rightarrow 0} \int_M |dP_{R,\varepsilon}| |d^* \alpha| |\bar{\beta}| d\mu(x) \\ &\leq \frac{1}{R} \int_M |Q^{-1/2} d^* \alpha| |\bar{\beta}| d\mu(x) \leq \frac{\|Q^{-1/2} d^* \alpha\| \|\beta\|}{R}. \end{aligned}$$

In the second inequality in (26) we used the estimate (19). The last inequality in (26) follows from Lemma 1.

Analogously, one can estimate the other terms in the right-hand side of (24). That proves (21) and Theorem A.

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