

THE EXPLICIT SOLUTION OF A DIFFUSION EQUATION WITH SINGULARITY

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ABSTRACT. We give the explicit solution of the Cauchy problem for the diffusion equation with a singular term:

$$\begin{aligned} (\partial/\partial t)u &= (\partial/\partial x)^2 u - (k/x^2)u, \quad t > 0, \quad x \in \mathbf{R}^1; \\ u(0, x) &= f(x), \quad x \in \mathbf{R}^1, \end{aligned}$$

where $k > -1/4$. We construct the solution on the basis of a generalization of the Fourier transform. We next show that the solution is expressed by an analytic semigroup, and examine smoothness of $x \mapsto u(t, x)$ and continuity of $x \mapsto u(t, x)/x^\beta$ ($\beta > 0$).

1. THE MAIN RESULT

In this paper we give the explicit solution of the Cauchy problem for the diffusion equation with a singular term:

$$(1.1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{k}{x^2} u, \quad t > 0, \quad x \in \mathbf{R}^1;$$

$$(1.2) \quad u(0, x) = f(x), \quad x \in \mathbf{R}^1,$$

where $k > -1/4$. We next show that the solution is expressed by an analytic semigroup, and examine smoothness of $x \mapsto u(t, x)$ and continuity of $x \mapsto u(t, x)/x^\beta$ ($\beta > 0$).

Let $f_+(x) = \{f(x) + f(-x)\}/2$, $f_-(x) = \{f(x) - f(-x)\}/2$, and let I_ν be a modified Bessel function, for which we refer the reader to the books [E] and [OK]. Our main result is the following.

Theorem 1.1. *Let $f \in L^2(\mathbf{R}^1)$. Then the explicit solution of the Cauchy problem (1.1), (1.2) is given by*

$$\begin{aligned} u(t, x) = \int_{\mathbf{R}^1} \frac{\sqrt{|x\xi|}}{4t} e^{-(x^2+\xi^2)/(4t)} \left\{ I_{a-1/2} \left(\frac{|x\xi|}{2t} \right) f_+(\xi) \right. \\ \left. + \operatorname{sgn}(x\xi) I_{b+1/2} \left(\frac{|x\xi|}{2t} \right) f_-(\xi) \right\} d\xi \end{aligned}$$

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for a.a. $x \in \mathbf{R}^1$, where $a, b > -1/2$ and

$$a(a - 1) = b(b + 1) = k.$$

Remark. When $k = 0$, (1.1) is reduced to the usual diffusion equation. Indeed, setting $a = b = 0$ in the explicit solution above and noting the formulas $I_{-1/2}(z) = \sqrt{2/(\pi z)} \cosh(z)$ and $I_{1/2}(z) = \sqrt{2/(\pi z)} \sinh(z)$, we obtain for a.a. $x \in \mathbf{R}^1$,

$$u(t, x) = \int_{\mathbf{R}^1} \frac{1}{\sqrt{4\pi t}} e^{-(x^2+\xi^2)/(4t)} \left\{ \cosh\left(\frac{x\xi}{2t}\right) f_+(\xi) + \sinh\left(\frac{x\xi}{2t}\right) f_-(\xi) \right\} d\xi$$

and hence arrive at the well-known formula represented by the Gaussian kernel:

$$(1.3) \quad u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}^1} e^{-(x-\xi)^2/(4t)} f(\xi) d\xi.$$

In what follows we therefore confine ourselves to the case where $k > -1/4$ and $k \neq 0$.

2. CONSTRUCTION OF THE SOLUTION

In this section, using a generalization of the Fourier transform, we construct the explicit solution of (1.1), (1.2) in the case where $k > -1/4$ (and $k \neq 0$) and give the proof of Theorem 1.1. The classical formula (1.3) for $k = 0$ is derived, for example, by means of the Fourier transform. Our generalized transform B_c ($c > -1/2$) has been introduced by Ohnuki and S. Watanabe [OW, Theorems 3.2 and 3.4] as a unitary operator from $L^2(\mathbf{R}^1)$ onto itself. We outline it for the reader's convenience. We have

$$(2.1) \quad \begin{aligned} B_c u(y) &= \text{l. i. m.}_{L \rightarrow \infty} \int_{-L}^L \overline{\varphi_c(xy)} u(x) dx, \\ B_c^* u(x) &= \text{l. i. m.}_{L \rightarrow \infty} \int_{-L}^L \varphi_c(xy) u(y) dy \end{aligned}$$

for $u \in L^2(\mathbf{R}^1)$. Here the function φ_c given by

$$(2.2) \quad \varphi_c(xy) = \frac{\sqrt{|xy|}}{2} \{ J_{c-1/2}(|xy|) + i \operatorname{sgn}(xy) J_{c+1/2}(|xy|) \}, \quad c > -\frac{1}{2},$$

was obtained by Ohnuki and Kamefuchi [OK, pp. 289-296], where J_ν is a Bessel function. If $c = 0$, then $\varphi_{c=0}(xy) = e^{ixy}/\sqrt{2\pi}$. In this case B_c coincides with the Fourier transform, and so it can be regarded as a generalization of the Fourier transform.

For each $m = 0, 1, 2, \dots$, let

$$(2.3) \quad \mathcal{H}_c^m(\mathbf{R}^1) = \left\{ u \in L^2(\mathbf{R}^1) : \int_{\mathbf{R}^1} (1+y^2)^m |B_c u(y)|^2 dy < \infty \right\}, \quad c > -\frac{1}{2}.$$

The space of Sobolev type $\mathcal{H}_c^m(\mathbf{R}^1)$ is a Hilbert space with inner product

$$(u, v)_m = \int_{\mathbf{R}^1} (1+y^2)^m B_c u(y) \overline{B_c v(y)} dy$$

and norm $|u|_m = (u, u)_m^{1/2}$ [W2] (see also [W1]). Furthermore, in [W2, Lemma 2.4] it is shown that the operator $(-i\mathcal{D}_c)^m$ is self-adjoint on the set $\mathcal{H}_c^m(\mathbf{R}^1)$, where

$$(2.4) \quad \mathcal{D}_c = \frac{d}{dx} - \frac{c}{x} R, \quad Ru(x) = u(-x), \quad c > -\frac{1}{2}.$$

See also [WW, Theorem 1] and [OW, Theorem 2.1 and Proposition 4.2] for the self-adjointness of $-i\mathcal{D}_c$. This operator \mathcal{D}_c appears in a one-dimensional harmonic oscillator governed by Wigner’s commutation relations [Wi], and the expression (2.4) was derived by Yang [Y]. As is well known, multiplication by y is self-adjoint on the set $D(y) = \{u(y) : u, yu \in L^2(\mathbf{R}^1)\}$. With the aid of the transform B_c , the operators $(-i\mathcal{D}_c)^m$ and y^m are unitarily equivalent to each other [OW, Proposition 4.2]:

$$(2.5) \quad B_c (-i\mathcal{D}_c)^m B_c^* = y^m, \quad B_c \mathcal{H}_c^m(\mathbf{R}^1) = D(y^m).$$

For unitary equivalence, see e.g. Goldstein [G, pp. 94-95].

Remark. The transform B_c depends on the constant c , and so does $\mathcal{H}_c^m(\mathbf{R}^1)$. When $c = 0$, $B_{c=0}$ coincides with the Fourier transform and $\mathcal{H}_{c=0}^m(\mathbf{R}^1)$ turns out to be the usual Sobolev space $H^m(\mathbf{R}^1)$.

We now turn to the Cauchy problem (1.1), (1.2). Let us decompose $u(t, x)$ into the sum of the even part $u_+(t, x)$ and the odd part $u_-(t, x)$:

$$u(t, x) = u_+(t, x) + u_-(t, x),$$

where

$$u_{\pm}(t, x) = \frac{1}{2} \{ u(t, x) \pm Ru(t, x) \} = \frac{1}{2} \{ u(t, x) \pm u(t, -x) \}.$$

Then the problem (1.1), (1.2) is decomposed into

$$(2.6) \quad \frac{\partial u_+}{\partial t} = \left(\frac{\partial^2}{\partial x^2} - \frac{k}{x^2} \right) u_+, \quad u_+(0, x) = f_+(x);$$

$$(2.7) \quad \frac{\partial u_-}{\partial t} = \left(\frac{\partial^2}{\partial x^2} - \frac{k}{x^2} \right) u_-, \quad u_-(0, x) = f_-(x).$$

Using the equality $\mathcal{D}_c^2 = (\partial/\partial x)^2 - x^{-2}c(c - R)$, we can rewrite (2.6) and (2.7) as

$$(2.8) \quad \frac{du_+}{dt} = \mathcal{D}_a^2 u_+, \quad u_+(0) = f_+;$$

$$(2.9) \quad \frac{du_-}{dt} = \mathcal{D}_b^2 u_-, \quad u_-(0) = f_-$$

where $a, b > -1/2$, $a(a - 1) = b(b + 1) = k$ and $f \in L^2(\mathbf{R}^1)$. In order to solve (2.8) and (2.9), we have only to deal with

$$(2.10) \quad \frac{du}{dt} = \mathcal{D}_c^2 u, \quad u(0) = f \in L^2(\mathbf{R}^1), \quad c > -\frac{1}{2},$$

and then to discuss the even and odd parts of the solution u .

By (2.5), the transform B_c turns (2.10) into

$$\frac{dU}{dt} = -y^2 U, \quad U(0) = B_c f,$$

where $U = B_c u$. An elementary calculation gives $U(t, y) = e^{-ty^2} B_c f(y)$. $U(t, \cdot)$ evidently belongs to $D(y^m)$ for every positive integer m . Hence (2.5) implies that $u(t, \cdot)$ belongs to $\mathcal{H}_c^m(\mathbf{R}^1)$ (see also (2.3)). Thus, recalling (2.1), we have

$$u(t, x) = \text{l.i.m.}_{L \rightarrow \infty} \int_{-L}^L \varphi_c(xy) e^{-ty^2} B_c f(y) dy.$$

Since $e^{-ty^2} B_c f \in D(y)$, it follows from [OW, Lemma 3.3] that

(2.11)

$$u(t, x) = v(t, x) \quad (\text{a.a. } x \in \mathbf{R}^1), \quad \text{where } v(t, x) = \int_{\mathbf{R}^1} \varphi_c(xy) e^{-ty^2} B_c f(y) dy.$$

Lemma 2.1. *Let v be given by (2.11). Then*

$$v(t, x) = \int_{\mathbf{R}^1} f(\xi) \left\{ \int_{\mathbf{R}^1} \varphi_c(xy) \overline{\varphi_c(\xi y)} e^{-ty^2} dy \right\} d\xi.$$

Proof. Set $F_L(y) = \int_{-L}^L \overline{\varphi_c(\xi y)} f(\xi) d\xi$. Then, letting $L \rightarrow \infty$ gives

$$\begin{aligned} & \left| v(t, x) - \int_{\mathbf{R}^1} \varphi_c(xy) e^{-ty^2} F_L(y) dy \right| \\ & \leq \sqrt{\int_{\mathbf{R}^1} |\varphi_c(xy)|^2 e^{-2ty^2} dy} \|B_c f - F_L\| \rightarrow 0, \end{aligned}$$

where $\|\cdot\|$ denotes the norm of $L^2(\mathbf{R}^1)$. Fubini's theorem therefore implies

$$v(t, x) = \lim_{L \rightarrow \infty} \int_{-L}^L f(\xi) \left\{ \int_{\mathbf{R}^1} \varphi_c(xy) \overline{\varphi_c(\xi y)} e^{-ty^2} dy \right\} d\xi,$$

which proves the lemma. □

It remains to compute the integral

$$I(x, \xi) = \int_{\mathbf{R}^1} \varphi_c(xy) \overline{\varphi_c(\xi y)} e^{-ty^2} dy.$$

Lemma 2.2.

$$I(x, \xi) = \frac{\sqrt{|x\xi|}}{4t} e^{-(x^2+\xi^2)/(4t)} \left\{ I_{c-1/2} \left(\frac{|x\xi|}{2t} \right) + \text{sgn}(x\xi) I_{c+1/2} \left(\frac{|x\xi|}{2t} \right) \right\}.$$

Proof. Recalling (2.2), we obtain

$$\begin{aligned} I(x, \xi) &= \frac{\sqrt{|x\xi|}}{2} \int_0^\infty e^{-ty^2} y J_{c-1/2}(|x|y) J_{c-1/2}(|\xi|y) dy \\ &\quad + \text{sgn}(x\xi) \frac{\sqrt{|x\xi|}}{2} \int_0^\infty e^{-ty^2} y J_{c+1/2}(|x|y) J_{c+1/2}(|\xi|y) dy. \end{aligned}$$

An appeal to the formula [E, (23), p. 51] (for $\text{Re } \nu > -1$ and $|\arg a| < \pi/4$)

$$\int_0^\infty e^{-a^2 y^2} y J_\nu(py) J_\nu(qy) dy = \frac{1}{2a^2} e^{-(p^2+q^2)/(4a^2)} I_\nu \left(\frac{pq}{2a^2} \right)$$

completes the proof. □

From Lemmas 2.1 and 2.2, we thus find that the solution u of (2.10) is given by

$$u(t, x) = \int_{\mathbf{R}^1} \frac{\sqrt{|x\xi|}}{4t} e^{-(x^2+\xi^2)/(4t)} \left\{ I_{c-1/2} \left(\frac{|x\xi|}{2t} \right) + \operatorname{sgn}(x\xi) I_{c+1/2} \left(\frac{|x\xi|}{2t} \right) \right\} f(\xi) d\xi$$

for a.a. $x \in \mathbf{R}^1$. Recalling (2.8) and (2.9), we see that Theorem 1.1 is true.

Remark. If $-1/4 < k < 3/4$ and $k \neq 0$, then $a = -b, b + 1$. Setting $a = -b, a = b + 1$ in (2.8), we have the following two equations:

$$\frac{du_+}{dt} = \mathcal{D}_{-b}^2 u_+, \quad \frac{du_+}{dt} = \mathcal{D}_{b+1}^2 u_+.$$

3. PROPERTIES OF THE SOLUTION

In this section we show that each of $u_{\pm}(t, \cdot)$ in Theorem 1.1 is expressed by an analytic semigroup, and examine smoothness of $x \mapsto u_{\pm}(t, x)$ and continuity of $x \mapsto u_{\pm}(t, x)/x^{\beta}$ ($\beta > 0$). To begin with we try to construct the solution of (1.1), (1.2) by the formula $u_{\pm}(t, x) = \exp(t \mathcal{D}_c^2) f_{\pm}(x)$ based on the abstract theory of semigroup generation:

Lemma 3.1. *The operator \mathcal{D}_c^2 generates an analytic semigroup $\{\exp(t \mathcal{D}_c^2) : t > 0\}$ on $L^2(\mathbf{R}^1)$.*

Proof. By (2.5) we have $B_c \mathcal{D}_c^2 B_c^* = -y^2$. The multiplication operator $-y^2$ generates the semigroup $\{\exp(-t y^2) : t > 0\}$, which is evidently an analytic semigroup. Since the transform B_c is unitary, the result follows. \square

From this lemma we obtain that the solution of (1.1), (1.2) with $f \in L^2(\mathbf{R}^1)$ is expressed by $u_{\pm}(t, x) = \exp(t \mathcal{D}_c^2) f_{\pm}(x)$, and $u_{\pm}(t, \cdot)$ belongs to $\mathcal{H}_c^{2m}(\mathbf{R}^1)$ for every $m = 1, 2, \dots$ and $t > 0$. Combining the inclusion $\exp(t \mathcal{D}_c^2) L^2(\mathbf{R}^1) \subset \mathcal{H}_c^{2m}(\mathbf{R}^1)$ and an embedding theorem of Sobolev type on $\mathcal{H}_c^{2m}(\mathbf{R}^1)$ (which was recently established by S. Watanabe [W2, Theorem 1.2]), we can study further properties of $u_{\pm}(t, \cdot)$. To this end we introduce the following spaces. For each $m = 0, 1, 2, \dots$, let

$$\mathcal{B}^m(\mathbf{R}^1) = \left\{ u \in C^m(\mathbf{R}^1) : \frac{d^k u}{dx^k} \ (k = 0, 1, \dots, m) \text{ are bounded on } \mathbf{R}^1 \right\},$$

$$\mathcal{F}^m(\mathbf{R}^1) = \left\{ u(x) : u, \frac{u}{x^k} \ (k = 1, 2, \dots, m) \text{ are continuous and bounded on } \mathbf{R}^1 \right\}.$$

As is well known, $\mathcal{B}^m(\mathbf{R}^1)$ is a Banach space with norm

$$\|u\|_{\mathcal{B}^m} = \max_{k=0,1,\dots,m} \left\{ \sup_{x \in \mathbf{R}^1} \left| \frac{d^k u}{dx^k}(x) \right| \right\},$$

and $\mathcal{F}^m(\mathbf{R}^1)$ is a Banach space with norm

$$\|u\|_{\mathcal{F}^m} = \max_{k=0,1,\dots,m} \left\{ \sup_{x \in \mathbf{R}^1} \left| \frac{u(x)}{x^k} \right| \right\},$$

as is shown in [W2, Lemma 2.1]. We describe below the statement of an embedding theorem of Sobolev type for the operator \mathcal{D}_c . It reveals the smoothness of $u \in \mathcal{H}_c^m(\mathbf{R}^1)$ and continuity of $x \mapsto u(x)/x^{\beta}$ ($\beta > 0$).

Lemma 3.2 ([W2, Theorem 1.2]). *Suppose $c > 1$. Then, for each $u \in \mathcal{H}_c^m(\mathbf{R}^1)$, there is an element $v \in \mathcal{B}^\alpha(\mathbf{R}^1) \cap \mathcal{F}^\beta(\mathbf{R}^1)$ such that $u(x) = v(x)$ (a.a. $x \in \mathbf{R}^1$), i.e.,*

$$\mathcal{H}_c^m(\mathbf{R}^1) \subset \mathcal{B}^\alpha(\mathbf{R}^1) \cap \mathcal{F}^\beta(\mathbf{R}^1),$$

where

$$\alpha = \begin{cases} m-1 & (c=2n), \\ \min(m-1, c-1) & (c=2n+1), \\ \min(m-1, [c]) & (\text{otherwise}), \end{cases}$$

$$\beta = \begin{cases} \min(m-1, c) & (c=2n), \\ \min(m-1, c-1) & (c=2n+1), \\ \min(m-1, [c]) & (\text{otherwise}) \end{cases}$$

with $n = 1, 2, 3, \dots$. Moreover, $|v|_{\mathcal{B}^\alpha} \leq c_1 |u|_m$ and $|v|_{\mathcal{F}^\beta} \leq c_2 |u|_m$, where c_1 and c_2 are positive constants that depend on m and c only.

Let $m = 1, 2, 3, \dots$. Then the solution $u_+(t, \cdot)$ in Theorem 1.1 (see also (2.8)) is in $\mathcal{H}_a^{2m}(\mathbf{R}^1)$, while the solution $u_-(t, \cdot)$ (see also (2.9)) is in $\mathcal{H}_b^{2m}(\mathbf{R}^1)$. We thus obtain the following result.

Theorem 3.3. (i) *Suppose $a > 1$. Then the even part $u_+(t, \cdot)$ of the solution u of the problem (1.1), (1.2) belongs to $\mathcal{B}^\alpha(\mathbf{R}^1) \cap \mathcal{F}^\beta(\mathbf{R}^1)$, where*

$$\alpha = \begin{cases} \text{any positive integer} & (a=2n), \\ a-1 & (a=2n+1), \\ [a] & (\text{otherwise}), \end{cases} \quad \beta = \begin{cases} a & (a=2n), \\ a-1 & (a=2n+1), \\ [a] & (\text{otherwise}) \end{cases}$$

with $n = 1, 2, 3, \dots$.

(ii) *Suppose $b > 1$. Then the odd part $u_-(t, \cdot)$ belongs to $\mathcal{B}^\alpha(\mathbf{R}^1) \cap \mathcal{F}^\beta(\mathbf{R}^1)$, where*

$$\alpha = \begin{cases} \text{any positive integer} & (b=2n), \\ b-1 & (b=2n+1), \\ [b] & (\text{otherwise}), \end{cases} \quad \beta = \begin{cases} b & (b=2n), \\ b-1 & (b=2n+1), \\ [b] & (\text{otherwise}) \end{cases}$$

with $n = 1, 2, 3, \dots$.

Remark. It is interesting that $u_+(t, \cdot)$ belongs to $C^\infty(\mathbf{R}^1)$ if $k = 2n(2n-1)$, and so does $u_-(t, \cdot)$ if $k = 2n(2n+1)$ for $n = 1, 2, 3, \dots$.

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