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VECTOR MOMENT PROBLEMS FOR RAPIDLY DECREASING SMOOTH FUNCTIONS OF SEVERAL VARIABLES

RICARDO ESTRADA

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Abstract. The existence of rapidly decreasing smooth solutions of moment problems for functions of several variables with values in a Fréchet space is obtained. It is shown that the corresponding results for functions with values in a general topological vector space do not hold.

1. Introduction

The problem of moments is an important mathematical problem which has attracted the attention of mathematicians for over a century. The moments have been shown to be of importance in several areas of the classical analysis [1], [14], while recent developments have shown that they also play a prominent role in areas of current interest such as the asymptotic expansion of generalized functions [4], [5], [7], [8], the theory of orthogonal polynomials [10], [12] and the theory of distributional solutions of differential equations [11], [16].

The aim of the present article is to study the vector moment problem for functions of several variables

$$\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x})\, \mathbf{x}^{\mathbf{k}}\, d\mathbf{x} = \mathbf{a}_{\mathbf{k}}, \qquad \mathbf{k} \in \mathbb{N}^n, \tag{1.1}$$

where $\{\mathbf{a}_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$ is a net of elements of a topological vector space E and where $\mathbf{f}\colon \mathbb{R}^n \to E$ is a rapidly decreasing smooth function. Notice the standard notation $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \dots x_n^{k_n}$ if $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{k} = (k_1, \dots, k_n)$.

We show that when E is a Fréchet space, then (1.1) has rapidly decreasing smooth solutions for arbitrary nets $\{\mathbf{a}_{\mathbf{k}}\}$. Actually, if $\mathcal{V}$ is an open cone in $\mathbb{R}^n$, then (1.1) has solutions with support in $\overline{\mathcal{V}}$. In the very important particular case when E is $\mathbb{R}$ or $\mathbb{C}$ we obtain that if $\{\mu_{\mathbf{k}}\}$ is an arbitrary net of real or complex numbers, then the moment problem

$$\int_{\mathcal{V}} \phi(\mathbf{x})\, \mathbf{x}^{\mathbf{k}}\, d\mathbf{x} = \mu_{\mathbf{k}}, \qquad \mathbf{k} \in \mathbb{N}^n,$$

has solutions $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subseteq \overline{\mathcal{V}}$.

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This work generalizes the results of A. J. Durán [2], who showed that in the one-variable scalar case the moment problem

$$\int_0^\infty \phi(x)\, x^k \, dx = \mu_k, \qquad k \in \mathbb{N},$$

has solutions $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \phi \subseteq [0, \infty)$ for each arbitrary sequence $\{\mu_k\}$. The methods employed here are based on those of [3].

We also consider the case when E is not a Fréchet space and give a counterexample showing that the existence of solutions for arbitrary nets does not hold for general topological vector spaces.

The plan of the article is as follows. In the second section we study the one-variable vector moment problem

$$\int_{-\infty}^\infty \mathbf{f}(x)\, x^k \, dx = \mathbf{a}_k, \qquad k \in \mathbb{N}, \tag{1.2}$$

where $\{\mathbf{a}_k\}$ is a sequence of elements of a Fréchet space E. In the next section we show that (1.2) might not have any solution if $E = \mathcal{D}(\mathbb{R})$, the standard space of test functions. In the fourth section we consider the case of vector functions of several variables. The use of tensor product considerations [9], [15] allow us to use an inductive argument.

2. One variable vector problems

Let E be a Fréchet space. Let $\| \; \|_1 \le \| \; \|_2 \le \| \; \|_3 \le \dots$ be a sequence of seminorms of E that generate its topology. Let $\{\mathbf{a}_k\}$ be an arbitrary sequence of elements of E. We wish to study the problem of finding a rapidly decreasing smooth function $\mathbf{f} \colon \mathbb{R} \to E$ such that

$$\int_{-\infty}^\infty \mathbf{f}(x)\, x^k \, dx = \mathbf{a}_k, \qquad k = 0, 1, 2, 3, \dots. \tag{2.1}$$

Notice that asking $\mathbf{f}$ to be a rapidly decreasing smooth function means that $\mathbf{f} \in \mathcal{S}(\mathbb{R}, E) \cong \mathcal{S}(\mathbb{R}) \mathbin{\widehat{\otimes}} E$. In general [15], $\mathbf{f}$ belongs to $\mathcal{S}(\mathbb{R}^n, E)$ if and only if for each $\mathbf{k}, \mathbf{m} \in \mathbb{N}^n$ the set $\{\, \mathbf{x}^{\mathbf{k}} \mathbf{D}^{\mathbf{m}} \mathbf{f}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n \,\}$ is bounded in E, where $\mathbf{D}^{\mathbf{m}} = \partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n}$ for $\mathbf{m} = (m_1, \dots, m_n)$. For a Fréchet space, this means that

$$\|\mathbf{D}^{\mathbf{m}} \mathbf{f}(\mathbf{x})\|_q = O(|\mathbf{x}|^{-k}), \qquad \text{as } |\mathbf{x}| \to \infty,$$

for each $\mathbf{m} \in \mathbb{N}^n$ and $q, k \in \mathbb{N}$.

We shall also have use for the space $\mathcal{S}((0,\infty), E) \cong \mathcal{S}(0,\infty) \mathbin{\widehat{\otimes}} E$ consisting of the elements of $\mathcal{S}(\mathbb{R}, E)$ with support in $[0, \infty)$.

Let us start by recalling the result of A. J. Durán [2].

Theorem 2.1. *Let $\{\mu_k\}$ be an arbitrary sequence of real or complex numbers. Then there exists $\phi \in \mathcal{S}(0,\infty)$ such that*

$$\int_0^\infty \phi(x)\, x^k \, dx = \mu_k,$$

for each $k \in \mathbb{N}$. □

Using this result, we can find functions $\phi_k \in \mathcal{S}(0, \infty)$ such that

$$\int_0^\infty \phi_k(x)\, x^n \, dx = \delta_{k,n}, \qquad n = 1, 2, 3, \dots, \tag{2.2}$$

where $\delta_{k,n} = 0$, $k \neq n$, $\delta_{n,n} = 1$, is the Kronecker delta. One is tempted to try to solve (2.1) by setting $\mathbf{f}(x) = \sum_{k=0}^{\infty} \phi_k(x)\mathbf{a}_k$. However, this series could be divergent and even if convergent the sum might not belong to $\mathcal{S}((0,\infty), E)$. But, as we show, a suitable modification of the series will do the job.

Observe that if $\phi_k(x)$ is a solution of (2.2), then $\lambda^{k+1}\phi_k(\lambda x)$ is also a solution for any $\lambda > 0$.

Lemma 2.2. *Let $\{\mathbf{a}_k\}$ be an arbitrary sequence of elements of the Fréchet space E. Then there exists a bounded, continuous and rapidly decreasing at infinity function $\mathbf{F}\colon (0,\infty) \to E$ such that*

$$\int_0^{\infty} \mathbf{F}(x)\, x^k \, dx = \mathbf{a}_k, \qquad k \in \mathbb{N}.$$

Proof. Let $\| \; \|_1 \leq \| \; \|_2 \leq \| \; \|_3 \leq \dots$ be a sequence of seminorms that generate the topology of E. Set $Q_n = \|\mathbf{a}_n\|_n$. Let $\phi_k \in \mathcal{S}(0,\infty)$ be a solution of (2.2). Then there exist constants $K_{n,r}$ such that

$$|\phi_n(x)| \leq \frac{K_{n,r}}{x^r}, \qquad x > 0.$$

We may suppose $K_{n,0} \leq K_{n,1} \leq K_{n,2} \leq \dots$. If $0 < \lambda < 1$ and $r \leq n$, then

$$|\lambda^{n+1}\phi_n(\lambda x)| \leq \lambda K_{n,n} x^{-r}, \qquad x > 0.$$

Choose the sequence $\{\lambda_n\}$ with $0 < \lambda_n < 1$ such that

$$\sum_{n=1}^{\infty} Q_n K_{n,n} \lambda_n < \infty,$$

and define

$$\mathbf{F}(x) = \sum_{n=0}^{\infty} \lambda_n^{n+1} \phi_n(\lambda_n x)\, \mathbf{a}_n. \tag{2.3}$$

Let $q, r \in \mathbb{N}$. Set $p = \max\{q, r\}$. Then

$$\sum_{n=0}^{\infty} \|\lambda_n^{n+1}\phi_n(\lambda_n x)\, \mathbf{a}_n\|_q \leq \left[\sum_{n=0}^{p-1} \lambda_n^{n+1-r} K_{n,r} \|\mathbf{a}_n\|_q + \sum_{n=p}^{\infty} Q_n K_{n,n} \lambda_n \right] x^{-r}.$$

The convergence of (2.3) for $x > 0$ as well as the continuity, boundedness and rapid decay at infinity of $\mathbf{F}(x)$ follow from this inequality. ☐

Lemma 2.3. *Let $\mathbf{F}\colon (0,\infty) \to E$ be bounded, continuous and of rapid decay at infinity. Let $\phi \in \mathcal{S}(0,\infty)$. Then their convolution, $\phi * \mathbf{F}$, given by*

$$(\phi * \mathbf{F})(x) = \int_0^x \phi(t)\mathbf{F}(x-t)\, dt, \qquad x > 0, \tag{2.4}$$

belongs to $\mathcal{S}((0,\infty), E)$.

Proof. If $\mathbf{F}$ and ϕ are regarded as functions on $\mathbb{R}$, with support in $[0, \infty)$, then their convolution $\phi * \mathbf{F}$ reduces to (2.4) if $x > 0$, vanishes if $x < 0$ and is smooth in $\mathbb{R}$ since ϕ is smooth. It remains to show that $(\phi * \mathbf{F})^{(k)} = \phi^{(k)} * \mathbf{F}$ is of rapid decay at infinity for each $k \in \mathbb{N}$. Let $q, r \in \mathbb{N}$, $r \geq 1$. Then there are constants M_1, M_2 such that

$$\begin{aligned} |\phi(x)| &\leq M_1 \min\{1, x^{-r-1}\}, \qquad x > 0, \\ \|\mathbf{F}(x)\|_q &\leq M_2 \min\{1, x^{-r-1}\}, \qquad x > 0. \end{aligned}$$

It follows that if $x > 0$,

$$\begin{aligned} \left\| \int_0^x \phi(t)\mathbf{F}(x-t)\, dt \right\|_q &\leq \int_0^x |\phi(t)|\, \|\mathbf{F}(x-t)\|_q \, dt \\ &\leq \int_0^{x/2} \frac{M_1 M_2\, dt}{(x-t)^{r+1}} + \int_{x/2}^x \frac{M_1 M_2\, dt}{t^{r+1}} \\ &\leq \frac{2M_1 M_2 (2^r - 1)}{r} x^{-r}, \end{aligned}$$

so that

$$(\phi * \mathbf{F})(x) = O(x^{-r}) \qquad \text{as } x \to \infty,$$

as required. □

Theorem 2.4. *Let* $\{\mathbf{a}_k\}$ *be an arbitrary sequence of the Fréchet space* E*. Then there exists* $\mathbf{f} \in \mathcal{S}((0,\infty), E)$ *such that*

$$\int_0^\infty \mathbf{f}(x)\, x^k \, dx = \mathbf{a}_k, \qquad k \in \mathbb{N}.$$

Proof. Let $\phi \in \mathcal{S}(0, \infty)$ be any function with $c_0 = \int_0^\infty \phi(x)\, dx \neq 0$. Define

$$c_k = \frac{1}{k!} \int_0^\infty \phi(x)\, x^k \, dx,$$

the normalized moments of ϕ, and define the sequence $\{\mathbf{b}_k\}$ of E by the recursion relation

$$\mathbf{b}_k = c_0^{-1} \left[\frac{\mathbf{a}_k}{k!} - \sum_{j=0}^{k-1} c_{k-j} \mathbf{b}_j \right], \qquad k \geq 0.$$

Using Lemma 2.2, we can find $\mathbf{F}\colon (0,\infty) \to E$ bounded, continuous and of rapid decay at infinity such that

$$\int_0^\infty \mathbf{F}(x)\, x^k \, dx = k!\, \mathbf{b}_k, \qquad k \geq 0.$$

Let $\mathbf{f} = \phi * \mathbf{F}$. Then $\mathbf{f} \in \mathcal{S}((0,\infty), E)$ by Lemma 2.3, while its moments can be computed as

$$\begin{aligned}
\int_0^\infty \mathbf{f}(x)\, x^k\, dx &= \int_0^\infty (\phi * \mathbf{F})(x)\, x^k\, dx \\
&= \int_0^\infty \int_0^\infty \phi(x)\mathbf{F}(t)\,(x+t)^k\, dx\, dt \\
&= \sum_{j=0}^{k} \binom{k}{j} \int_0^\infty \phi(x)\, x^j\, dx \int_0^\infty \mathbf{F}(t)\, t^{k-j}\, dt \\
&= k! \sum_{j=0}^{k} c_j \mathbf{b}_{k-j} \\
&= \mathbf{a}_k.
\end{aligned}$$

□

3. A counterexample

In this section we show that Theorem 2.4 does not hold, in general, if E is not a Fréchet space.

Let us take $E = \mathcal{D}(\mathbb{R}) = \{\, \phi \in C^\infty(\mathbb{R}) : \operatorname{supp}\phi \text{ is compact}\,\}$, the space of standard test functions [13]. A net $\{\phi_\sigma\}$ of $\mathcal{D}(\mathbb{R})$ converges to $\phi \in \mathcal{D}(\mathbb{R})$ if there is a fixed compact $K \subseteq \mathbb{R}$ and an index σ_0 such that $\operatorname{supp}\phi_\sigma \subseteq K$ for $\sigma \geq \sigma_0$ and $\{\phi_\sigma^{(k)}\}$ converges uniformly to $\phi^{(k)}$ for each $k = 0, 1, 2, \ldots$.

If $\phi\colon X \to \mathcal{D}(\mathbb{R})$ is a function, we write its values as $\phi(x,t) = \phi_x(t)$, $x \in X$, $t \in \mathbb{R}$.

Lemma 3.1. *Let X be a compact space and let $\phi\colon X \to \mathcal{D}(\mathbb{R})$ be continuous. Then there is a fixed compact $K \subseteq \mathbb{R}$ such that* $\operatorname{supp}\phi_x \subseteq K$ *for each $x \in X$.*

Proof. If not, we can find a sequence $\{x_n\}$ of X such that $\operatorname{supp}\phi_{x_n} \cap (\mathbb{R}\setminus[-n,n]) \neq \emptyset$. By taking a subsequence, if necessary, we may assume $\{x_n\}$ convergent. Then ϕ_{x_n} is convergent in $\mathcal{D}(\mathbb{R})$ and thus there exists $N > 0$ such that $\operatorname{supp}\phi_{x_n} \subseteq [-N, N]$ for each n in $\mathbb{N}$: a contradiction. □

Let now $\phi(x,t) = \phi_x(t)$ be a kernel of $\mathcal{S}(\mathbb{R}, \mathcal{D}(\mathbb{R}))$. Then ϕ_x vanishes as $|x| \to \infty$ and then ϕ can be continuously extended to the one point compactification $\mathbb{R} \cup \{\infty\}$ by setting $\phi_\infty = 0$. By the lemma, there is a fixed compact K such that $\operatorname{supp}\phi_x \subseteq K$ for each x in $\mathbb{R}$. Consequently all the moments

$$\psi_k(t) = \int_{-\infty}^\infty \phi(x,t)\, x^k\, dx$$

are also supported on K. Therefore, we have the following result.

Proposition 3.2. *Let $\{\psi_k\}$ be a sequence of $\mathcal{D}(\mathbb{R})$ such that $\bigcup_{k=0}^\infty \operatorname{supp}\psi_k$ is not bounded. Then the moment problem*

$$\int_{-\infty}^\infty \phi(x,t)\, x^k\, dx = \psi_k(t), \qquad k = 0, 1, 2, 3, \ldots$$

has no solution ϕ in $\mathcal{S}(\mathbb{R}, \mathcal{D}(\mathbb{R}))$. □

4. Moment problems for functions of several variables

We now consider the vector moment problem for functions of several variables with values on a Fréchet space.

In what follows we use the notation $\mathcal{S}(\mathcal{V}, E)$ for the space of functions of $\mathcal{S}(\mathbb{R}^n, E)$ whose support is contained in $\overline{\mathcal{V}}$ if $\mathcal{V}$ is an open subset of $\mathbb{R}^n$.

Theorem 4.1. *Let E be a Fréchet space and let $\{\mathbf{a_k}\}_{\mathbf{k}\in\mathbb{N}^n}$ be a net of elements of E. Then there exists $\mathbf{f} \in \mathcal{S}((0,\infty)^n, E)$ such that*

$$\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x})\,\mathbf{x}^{\mathbf{k}}\,d\mathbf{x} = \mathbf{a_k}, \qquad \mathbf{k} \in \mathbb{N}^n.$$

Proof. We apply induction on n, the number of variables of the function $\mathbf{f}$. If $n = 1$, then the existence of solutions of the moment problem is Theorem 2.4. Let us suppose the existence of solutions holds for functions of $n-1$ variables. If $\mathbf{k} \in \mathbb{N}^n$, write $\mathbf{k} = (\mathbf{j}, i)$, where $\mathbf{j} \in \mathbb{N}^{n-1}$ and $i \in \mathbb{N}$. Observe that the space $\mathcal{S}((0,\infty)^{n-1}, E) \cong \mathcal{S}(0,\infty)^{n-1} \mathbin{\widehat{\otimes}} E$ is a Fréchet space [15]. Thus, by the induction hypotesis, for each $i \in \mathbb{N}$ we can find a solution $\mathbf{g}_i \in \mathcal{S}((0,\infty)^{n-1}, E)$ of the moment problem

$$\int_{\mathbb{R}^{n-1}} \mathbf{g}_i(\mathbf{z})\,\mathbf{z}^{\mathbf{j}}\,d\mathbf{z} = \mathbf{a}_{(\mathbf{j},i)}, \qquad \mathbf{j} \in \mathbb{N}^{n-1}.$$

Next, by Theorem 2.4, we can find $\mathbf{h} \in \mathcal{S}((0,\infty), \mathcal{S}((0,\infty)^{n-1}, E))$ such that

$$\int_0^\infty \mathbf{h}(t)\,t^i\,dt = \mathbf{g}_i, \qquad i \in \mathbb{N}.$$

If $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (\mathbf{z}, t)$, $\mathbf{z} \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$ and $\mathbf{f}(\mathbf{x}) = \mathbf{h}(t)(\mathbf{z})$.

Since $\mathcal{S}((0,\infty), \mathcal{S}((0,\infty)^{n-1}, E)) \cong \mathcal{S}((0,\infty)^n, E)$, it follows that $\mathbf{f} \in \mathcal{S}((0,\infty)^n, E)$. Also

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x})\,\mathbf{x}^{\mathbf{k}}\,d\mathbf{x} &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathbf{h}(t)(\mathbf{z})\,t^i \mathbf{z}^{\mathbf{j}}\,dt\,d\mathbf{z} \\ &= \int_{\mathbb{R}^{n-1}} \mathbf{g}_i(\mathbf{z})\,\mathbf{z}^{\mathbf{j}}\,d\mathbf{z} \\ &= \mathbf{a_k},\end{aligned}$$

where $\mathbf{k} = (\mathbf{j}, i)$. ☐

A simple change of variables shows that the solution of the moment problem can be taken with support in an arbitrary cone with non-empty interior.

Theorem 4.2. *Let E be a Fréchet space, $\{\mathbf{a_k}\}_{\mathbf{k}\in\mathbb{N}^n}$ an arbitrary net of elements of E. Let $\mathcal{V}$ be an open cone in $\mathbb{R}^n$. Then there exists $\mathbf{f} \in \mathcal{S}(\mathcal{V}, E)$ such that*

$$\int_{\mathcal{V}} \mathbf{f}(\mathbf{x})\,\mathbf{x}^{\mathbf{k}}\,d\mathbf{x} = \mathbf{a_k}, \qquad \mathbf{k} \in \mathbb{N}^n. \tag{4.1}$$

Proof. Let $\mathbf{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear non-singular transformation with $\mathbf{T}(\mathcal{V}) \subseteq (0,\infty)^n$. If $\mathbf{g} \in \mathcal{S}((0,\infty)^n, E)$, then the function $\mathbf{f}$ defined by $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{T}\,\mathbf{x})$ belongs to $\mathcal{S}(\mathcal{V}, E)$. There is a simple relation between the moments of $\mathbf{f}$ and $\mathbf{g}$. Let

$$\mathcal{M}\big[\mathbf{f}(\mathbf{x}), p(\mathbf{x})\big] = \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}) p(\mathbf{x})\,d\mathbf{x},$$

be the moment function associated with $\mathbf{f}$; $\mathcal{M}$ is an operator from the space of polynomials in n variables to E. Clearly $\mathcal{M}\big[\mathbf{f}(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\big]$ are the moments of $\mathbf{f}$.

On the other hand, if $\{\mathbf{a_k}\}_{\mathbf{k}\in\mathbb{N}^n}$ is a net of elements of E, we can also define a moment function $\mathcal{N}\big[\{\mathbf{a_k}\}, p(\mathbf{x})\big]$ by requiring that $\mathcal{N}\big[\{\mathbf{a_k}\}, \mathbf{x^q}\big] = \mathbf{a_q}$ and extending by linearity. Clearly

$$\mathcal{N}\big[\{\mathbf{a_k}\}, p(\mathbf{x})\big] = \mathcal{M}\big[\mathbf{f}(\mathbf{x}), p(\mathbf{x})\big]$$

if $\mathbf{a_q} = \mathcal{M}\big[\mathbf{f}(\mathbf{x}), \mathbf{x^q}\big]$ for each $\mathbf{q} \in \mathbb{N}^n$.

If $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{T}\,\mathbf{x})$, then the moment functions $\mathcal{M}\big[\mathbf{f}(\mathbf{x}), -\big]$ and $\mathcal{M}\big[\mathbf{g}(\mathbf{x}), -\big]$ are related as

$$\mathcal{M}\big[\mathbf{g}(\mathbf{x}), p(\mathbf{x})\big] = \int_{\mathbb{R}^n} \mathbf{g}(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} = |\det \mathbf{T}| \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{u})p(\mathbf{T}\,\mathbf{u})\,d\mathbf{u},$$

or

$$\mathcal{M}\big[\mathbf{g}(\mathbf{x}), p(\mathbf{x})\big] = |\det \mathbf{T}|\,\mathcal{M}\big[\mathbf{f}(\mathbf{x}), p(\mathbf{T}\,\mathbf{x})\big]. \tag{4.2}$$

Therefore, if $\{\mathbf{a_k}\}$ is an arbitrary net in E, we can solve the problem (4.1) by defining $\mathbf{f} \in \mathcal{S}(\mathcal{V}, E)$ as $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{T}\,\mathbf{x})$, where $\mathbf{T}$ is a non-singular linear transformation with $\mathbf{T}(\mathcal{V}) \subseteq (0,\infty)^n$ and where $\mathbf{g} \in \mathcal{S}((0,\infty)^n, E)$ is a solution of the moment problem

$$\mathcal{M}\big[\mathbf{g}(\mathbf{x}), \mathbf{x^k}\big] = |\det \mathbf{T}|\,\mathcal{N}\big[\{\mathbf{a_q}\}, (\mathbf{T}\,\mathbf{x})^{\mathbf{k}}\big]. \tag{4.3}$$

Indeed, it follows by linearity from (4.3) that

$$\mathcal{M}\big[\mathbf{g}(\mathbf{x}), p(\mathbf{x})\big] = |\det \mathbf{T}|\,\mathcal{N}\big[\{\mathbf{a_q}\}, p(\mathbf{T}\,\mathbf{x})\big],$$

and, on comparing with (4.2), that

$$\mathcal{M}\big[\mathbf{f}(\mathbf{x}), p(\mathbf{x})\big] = \mathcal{N}\big[\{\mathbf{a_q}\}, p(\mathbf{x})\big],$$

so that, in particular,

$$\mathcal{M}\big[\mathbf{f}(\mathbf{x}), \mathbf{x^k}\big] = \mathbf{a_k},$$

as required. □

We remark that when $\mathcal{V}$ is not a cone, existence results of this kind are not to be expected. Actually, if $\overline{\mathcal{V}}$ is compact, even in the one-variable scalar case and even if the solutions are allowed to be distributions, the problem has a solution only if certain additional conditions are satisfied [6].

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Department of Mathematics, Texas A&M University, College Station, Texas 77843
Current address: P. O. Box 276, Tres Ríos, Costa Rica
E-mail address: restrada@cariari.ucr.ac.cr