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ON THE EXTENDED HILBERT'S INEQUALITY

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ABSTRACT. In this paper, it is shown that the extended Hilbert's inequality for double series can be refined by the aid of the Euler-Maclaurin summation formula. The extreme cases $p \to 1^+$ and $q \to +\infty$ are discussed.

1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\sum_{n=1}^{\infty} a_n^p < +\infty$ and $\sum_{n=1}^{\infty} b_n^q < +\infty$, then an extended Hilbert's inequality may be written in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \left(\pi/\sin\frac{\pi}{p}\right)\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

As is well known, the constant factor $\pi/\sin\frac{\pi}{p}$ contained in (1) is best possible. In other words, $\pi/\sin\frac{\pi}{p}$ cannot be replaced by any positive number smaller than it (cf. [1], [2]). But we may move the factor $\pi/\sin\frac{\pi}{p}$ of the right-hand side of (1) to the inside of the summation and write it in the following form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \left\{\sum_{n=1}^{\infty}(\pi/\sin\frac{\pi}{p} - \alpha_n(q))a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}(\pi/\sin\frac{\pi}{p} - \alpha_n(p))b_n^q\right\}^{\frac{1}{q}} \tag{2}$$

where $\alpha_n(r) \downarrow 0$ $(r = p, q)$. Clearly, it will offer a refined form of (1). In this paper it will be shown that we can take $\alpha_n(r) = \lambda/n^{1-\frac{1}{r}}$, where λ is a positive real number that is independent of r. Furthermore, we prove also that $\lambda = 1-\gamma$, where γ is the Euler constant.

Before proving our results we need to define some functions. Throughout this paper we assume that $x \in [1, +\infty)$ and $r \in (1, +\infty)$.

Let us define the following functions:

$$u(x) = x^{1-\frac{1}{r}} I(x), \tag{3}$$

where $I(x)$ is defined by

$$I(x) = \int_0^{\frac{1}{x}} \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{r}} dt, \tag{4}$$

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and

$$v(x) = \int_1^\infty \rho(t) F(x,t)\, dt, \tag{5}$$

where $\rho(t)$ and $F(x,t)$ are defined respectively by

$$\rho(t) = t - [t] - \frac{1}{2} \quad \text{and} \quad F(x,t) = \frac{(r+1)xt + x^2}{r(x+t)^2 t^{1+1/r}}.$$

For convenience we define

$$\lambda_r(x) = u(x) + v(x) - \frac{x}{2(x+1)} \tag{6}$$

where $u(x)$ and $v(x)$ are defined respectively by (3) and (5). Particularly, in the case $x = 1$, $\lambda_r(1)$ is denoted by $\lambda(r)$. We know from (6) that

$$\lambda(r) = u(1) + v(1) - \frac{1}{4}. \tag{7}$$

We will show that $\lambda(r)$ can be written in the form

$$\lambda(r) = J(r) + R(r), \tag{8}$$

where J and R are defined respectively by

$$J(r) = \int_0^1 \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{r}} dt - \frac{13r+2}{48r}$$

and

$$R(r) = \frac{\theta}{5760}\left(3 + \frac{20}{r} + \frac{18}{r^2} + \frac{4}{r^3}\right) \qquad (0 < \theta < 1).$$

2. Lemmas

The aim of the section is to prove the following inequalities are valid:

$$\lambda_r(x) \geq \lambda(r) > \lambda$$

where λ is an infimum of $\lambda(r)$.

Lemma 1. *Let $I(x)$ be the function defined by (4). Then*

$$I(x) \geq \frac{r(2r-1)x^{\frac{1}{r}}}{(r-1)((2r-1)x + r - 1)}. \tag{9}$$

Proof. Using integration by parts we obtain

$$I(x) = \frac{rx^{\frac{1}{r}}}{(r-1)(x+1)} + \frac{r}{r-1}K(x), \tag{10}$$

where $K(x)$ is defined by

$$K(x) = \int_0^{\frac{1}{x}} \frac{t^{1-\frac{1}{r}}}{(1+t)^2}\, dt.$$

Define the functions f and g respectively by

$$f(t) = \frac{1}{(1+t)^2}\left(\frac{1}{x}\right)^{1-\frac{1}{r}} \quad \text{and} \quad g(t) = (xt)^{1-\frac{1}{r}}, \qquad t \in \left[0, \frac{1}{x}\right].$$

Evidently $f(t)$ is nonnegative and monotone decreasing in $[0, \frac{1}{x}]$ and $g(t)$ satisfies the constraint $0 \le g(t) \le 1$. According to Steffensen's inequality we have

$$\int_{\frac{1}{x}-c}^{\frac{1}{x}} f(t)\,dt \le \int_0^{\frac{1}{x}} f(t)g(t)\,dt = K(x) \le \int_0^c f(t)\,dt, \tag{11}$$

where $c = \int_0^{\frac{1}{x}} g(t)\,dt = \int_0^{\frac{1}{x}} (xt)^{1-\frac{1}{r}}\,dt = \frac{r}{x(2r-1)}$. Hence

$$K(x) \ge \int_{\frac{1}{n}-c}^{\frac{1}{n}} \frac{1}{(1+t)^2}\left(\frac{1}{x}\right)^{1-\frac{1}{r}} dt = -\frac{x^{\frac{1}{r}}}{x+1} + \frac{x^{\frac{1}{r}}(2r-1)}{x(2r-1)+(r-1)}.$$

Substituting it in the second term of the right-hand side of (10) we obtain after simplification that (9) is valid. □

Lemma 2. *Let $\lambda_r(x)$ be the function defined by* (6). *Then*

$$\lambda_r(n) \ge \lambda(r), \qquad n \in N, \tag{12}$$

where $\lambda(r)$ is defined by (7).

Proof. At first, consider the function $u(x)$ defined by (3). Taking derivatives and after simplification we have

$$u'(x) = \frac{r-1}{r} x^{-\frac{1}{r}} I(x) - \frac{x^{1-\frac{1}{r}}}{1+x}.$$

By Lemma 1 we obtain easily the following inequality:

$$u'(x) \ge r/(x+1)((2r-1)x+(r-1)). \tag{13}$$

Define the functions F_1 and F_2 by

$$F_1(t) = \frac{r+1}{r(x+t)^2 t^{1/r}} \quad \text{and} \quad F_2(t) = \frac{2x}{(x+t)^3 t^{1/r}}, \qquad t \in [1, +\infty).$$

Obviously $F_i(t) \downarrow 0$ $(t \to +\infty)$ and after calculations $F_i''(t) > 0$. In the paper [4] it has been proved that

$$-\frac{1}{8}F_i(1) < \int_1^\infty \rho(t)F_i(t)\,dt < -\frac{1}{12}F_i\left(\frac{3}{2}\right), \qquad i = 1, 2.$$

Hence we obtain from (5) that

$$\begin{aligned} v'(x) &= \int_1^\infty \rho(t)\left(\frac{x+rt+t-xr}{r(x+t)^3 t^{1/r}}\right) dt \\ &= \int_1^\infty \rho(t)F_1(t)\,dt - \int_1^\infty \rho(t)F_2(t)\,dt \\ &> -\frac{1}{8}F(1) + \frac{1}{12}F_2\left(\frac{3}{2}\right) \\ &= -\frac{r+1}{8r(x+1)^2} + \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}}. \end{aligned} \tag{14}$$

We can obtain from (13) and (14) that

$$\begin{aligned} \lambda_r'(x) &= u'(x) + v'(x) - \frac{1}{2(x+1)^2} \\ &> \frac{(-2r^2+3r+1)x + (3r^2+4r+1)}{8r(x+1)^2((2r-1)x+(r-1))} + \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}}. \end{aligned}$$

By direct computations we have the following conclusions (see Notes at the end of this paper):

When $r \geq 4, (\frac{2}{3})^{\frac{1}{r}} > \frac{9}{10}$, $\lambda_r'(x) > 0$ is true. And when $1 < r < 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{2}{3}$, $\lambda_r'(x) > 0$ is also true. This implies that $\lambda_r(x)$ is monotone increasing. Whence (12) is valid. □

Lemma 3. *Let $\lambda(r)$ be the function defined by* (8). *Then if $\lambda = \inf\{\lambda(r)\}$ we have $\lim_{r\to\infty} \lambda(r) = \lambda$.*

Proof. Evidently the function $J(r)$ is continuously differentiable in $(1, +\infty)$. Hence

$$J'(r) = \frac{1}{r^2}\int_0^1 \frac{\ln t}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{r}} dt + \frac{1}{24r^2}. \tag{15}$$

Substituting $t = e^{-y}$ in (15) we obtain easily that

$$\begin{aligned} J'(r) &= -\frac{1}{r^2}\int_0^{+\infty} \frac{ye^{-\alpha y}}{1+e^{-y}}\,dy + \frac{1}{24r^2} \\ &< -\frac{1}{2r^2}\int_0^{+\infty} ye^{-\alpha y}\,dy = -\frac{1}{2(r-1)^2} + \frac{1}{24r^2} < 0, \end{aligned}$$

where $\alpha = 1 - \frac{1}{r}$. Hence the function $J(r)$ is monotone decreasing. Clearly $R(r)$ is also monotone decreasing. Thus

$$\lambda = \inf\{\lambda(r)\} = \lim_{r\to\infty} \lambda(r). \quad \square$$

By Lemma 3, we obtain at once the following results:

$$\lambda(r) > \lambda = \ln 2 - \frac{13}{48} + \frac{\theta}{1920} \qquad (0 < \theta < 1). \tag{16}$$

3. Main results

Theorem 1. *Let $q \geq p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < \sum_{n=1}^{\infty} a_n^p < +\infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < +\infty$, then*

(17)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \lambda/n^{\frac{1}{p}}\right) a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{n}{p}} - \lambda/n^{\frac{1}{q}}\right) b_n^q\right\}^{\frac{1}{q}},$$

where $\lambda = 1 - \gamma$ and γ is the Euler constant. λ is the largest constant that keeps (17) *valid and is independent of r $(r = p, q)$.*

Proof. We may apply Hölder's inequality to estimate the left-hand side of (17) as follows:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m+n)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}\cdot\frac{b_n}{(m+n)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}\\ &\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right\}^{\frac{1}{p}}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right)a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)b_n^q\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{n=1}^{\infty}\omega_q(n)a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega_p(n)b_n^q\right\}^{\frac{1}{q}},\end{aligned}$$

where $\omega_r(n)$ $(r=p,q)$ is defined by

$$\omega_r(n)=\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{r}}.$$

Applying the Euler-Maclaurin summation formula to $\omega_r(n)$ and using the relation $\sin\frac{\pi}{p}=\sin\frac{\pi}{q}$, we obtain

$$\begin{aligned}\omega_r(n) &= \int_1^{\infty}g(t)\,dt+\frac{1}{2}g(1)+\int_1^{\infty}\rho(t)g'(t)\,dt\\ &= \int_0^{\infty}g(t)\,dt-\int_0^{1}g(t)\,dt+\frac{1}{2}g(1)+\int_1^{\infty}\rho(t)g'(t)\,dt,\end{aligned}$$

where the function g is defined by

$$g(t)=\frac{1}{n+t}\left(\frac{n}{t}\right)^{\frac{1}{r}},\qquad t\in(0,+\infty).$$

Note that

$$\int_0^{\infty}g(t)\,dt=\pi/\sin\frac{\pi}{p},\qquad \int_0^{1}g(t)\,dt=\int_0^{\frac{1}{n}}\frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{r}}\,dt=u(n)/n^{1-\frac{1}{r}}$$

and

$$\int_1^{\infty}\rho(t)g^1(t)\,dt=v(n)/n^{1-\frac{1}{r}},$$

where $u(x)$ and $v(x)$ are the functions defined by (3) and (5) respectively. Hence

$$\begin{aligned}\omega_r(n) &= \pi/\sin\frac{\pi}{p}-\left(u(n)+v(n)-\frac{n}{2(n+1)}\right)/n^{1-\frac{1}{r}}\\ &= \pi/\sin\frac{\pi}{p}-\lambda_r(n)/n^{1-\frac{1}{r}},\end{aligned}\tag{18}$$

where $\lambda_r(n)$ is the function defined by (6).

In view of (12) we have

$$\omega_r(n)\le\pi/\sin\frac{\pi}{p}-\lambda(r)/n^{1-\frac{1}{r}}.\tag{19}$$

When $n = 1$ it follows from (18) that

$$\lambda(r) = \lambda_r(1) = \pi/\sin\frac{\pi}{p} - \omega_r(1).$$

Applying the Euler-Maclaurin summation formula to $\omega_r(1)$ we have

$$\begin{aligned}\omega_r(1) &= \sum_{m=1}^{\infty}\frac{1}{1+m}\left(\frac{1}{m}\right)^{\frac{1}{r}} = \int_1^{\infty} f(t)\,dt + \frac{1}{2}f(1) + \sum_{k=1}^{s-1} - \rho_s \\ &= \int_0^{\infty} f(t)\,dt - \int_0^1 f(t)\,dt + \frac{1}{2}f(1) + \sum_{k=1}^{s-1} - \rho_s \\ &= \pi/\sin\frac{\pi}{p} - \int_0^1 f(t)\,dt + \frac{1}{4} + \sum_{k=1}^{s-1} - \rho_s.\end{aligned}$$

Hence the term $\lambda(r)$ that appears in (19) can be written in the form

$$\lambda(r) = \int_0^1 f(t)\,dt - \frac{1}{4} - \sum_{k=1}^{s-1} + \rho_s, \tag{20}$$

where $f(t) = \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{r}}$, $\sum_{k=1}^{s-1} = \sum_{k=1}^{s-1}\frac{B_2k}{(2k)!}f_{(1)}^{(2k-1)}$ and the B_j's are the Bernoulli numbers, viz. $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ etc., and ρ_s is the remainder of the form

$$\rho_s = \frac{B_{2s}\theta}{(2s)!}f^{(2s-1)}(1) \qquad (0 < \theta < 1).$$

For $s = 2$ we obtain (8) from (20). By virtue of Lemma 3 and (19) we get that

$$\omega_r(n) < \pi/\sin\frac{\pi}{p} - \lambda/n^{1-\frac{1}{r}}. \tag{21}$$

It remains to show that $\lambda = 1 - \gamma$, where γ is the Euler constant. For $n = 1$, using the Euler-Maclaurin summation formula we obtain from (18) that

$$\begin{aligned}\lambda(r) &= \pi/\sin\frac{\pi}{p} - \left\{\sum_{m=1}^{k-1}\frac{1}{1+m}\left(\frac{1}{m}\right)^{\frac{1}{r}} + \sum_{m=k}^{\infty}\frac{1}{1+m}\left(\frac{1}{m}\right)^{\frac{1}{r}}\right\} \\ &= \pi/\sin\frac{\pi}{p} - \left\{\sum_{m=1}^{k-1}\frac{1}{1+m}\left(\frac{1}{m}\right)^{\frac{1}{r}} + \int_k^{\infty} f(t)\,dt + \frac{1}{2}f(k) - \frac{\theta}{12}f'(k)\right\} \\ &= \int_0^k f(t)\,dt - \sum_{m=1}^{k-1}\frac{1}{1+m}\left(\frac{1}{m}\right)^{\frac{1}{r}} - \frac{1}{2}f(k) + \frac{\theta}{12}f'(k) \qquad (0 < \theta < 1),\end{aligned} \tag{22}$$

where $f(t) = \frac{1}{1+t}(\frac{1}{t})^{\frac{1}{r}}$. In accordance with the definition of the Euler constant γ, i.e.

$$\sum_{m=0}^{k-1}\frac{1}{1+m} = \gamma + \ln(k-1) + \varepsilon_{k-1} \qquad (\varepsilon_{k-1} \to 0,\ \text{if } k \to +\infty)$$

and by Lemma 3 we obtain from (22)

$$\begin{aligned}\lambda = \lim_{r\to\infty}\lambda(r) &= \int_0^k \frac{1}{1+t}\,dt - \sum_{m=1}^{k-1}\frac{1}{1+m} - \frac{1}{2(1+k)} - \frac{\theta}{12(1+k)^2} \\ &= 1 - \gamma + \Delta R,\end{aligned}$$

where ΔR is the error of the form

$$\Delta R = \ln\frac{k+1}{k-1} - \varepsilon_{k-1} - \frac{1}{2(1+k)} - \frac{\theta}{12(1+k)^2} \qquad (0 < \theta < 1).$$

This implies that the bigger we take the value of k, the smaller the value of $|\Delta R|$. Let $k \to +\infty$. Then we obtain that $\lambda = 1 - \gamma$.

Based on Lemma 3 and (21), it follows that λ is the largest constant that keeps (17) valid and is independent of r $(r = p, q)$.

Thus we have completed the proof of the theorem. □

The value of λ is given numerically as follows:

$$\lambda = 0.422784335098467\ldots.$$

In particular, in the case $r = 2$, it follows from (8) that

$$\lambda(2) = J(2) + R(2) = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320}.$$

In view of (19) we have

$$\omega_2(n) \leq \pi - \lambda(2)/\sqrt{n}.$$

Therefore we obtain a sharp result of Hilbert's inequality.

Theorem 2. *If* $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^2 < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}(\pi - \alpha/\sqrt{n})a_n^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}(\pi - \alpha/\sqrt{n})b_n^2\right\}^{\frac{1}{2}},$$

where $\alpha = \frac{\pi}{2} - \frac{7}{24} + \frac{\theta}{320}$ $(0 < \theta < 1)$.

Finally, the extreme cases $p \to 1^+$ and $q \to +\infty$ are discussed. Note that $\frac{1}{p} + \frac{1}{q} = 1$ and $q \geq p > 1$. In the paper [3] it has been proved that $\lambda(p) > \frac{1}{p-1}$, where $\lambda(p)$ is defined by (20). Now we may prove that $\lambda(p) \sim \frac{1}{p-1}$ when $p \to 1^+$. In fact, for $r = p$ and $x = 1$ we consider the function defined by (4) and denote it by $h(p)$. We have

$$h(p) = \int_0^1 \frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{p}} dt.$$

From (10) we obtain

$$h(p) = \frac{p}{2(p-1)} + \frac{p}{p-1}k(1), \quad \text{where } k(1) = \int_0^1 \frac{t^{1-\frac{1}{p}}}{(1+t)^2}\,dt.$$

Use (11) to estimate $k(1)$. When $x = 1$ we have

$$c = \int_0^1 g(t)\,dt = \int_0^1 t^{1-\frac{1}{p}}\,dt = \frac{p}{2p-1},$$

$$\int_{1-c}^1 f(t)\,dt = \int_{1-c}^1 \frac{dt}{(1+t)^2} = \frac{p}{2(3p-2)}$$

and $\int_0^c f(t)\,dt = \frac{p}{3p-1}$. Hence

$$\frac{p}{2(3p-2)} \leq k(1) \leq \frac{p}{3p-1}.$$

Since $\lim_{p\to 1^+} \frac{p}{2(3p-2)} = \lim_{p\to 1^+} \frac{p}{3p-1} = \frac{1}{2}$, it follows that $\lim_{p\to 1^+} k(1) = \frac{1}{2}$. Whence $\lim_{p\to 1^+}(p-1)h(p) = 1$. It follows from (8) that $\lim_{p\to 1^+} \lambda(p)/\frac{1}{p-1} = 1$.

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Notes

$\lambda_r'(x) > g(x)$, where

$$g(x) = \frac{(-2r^2+3r+1)x + (3r^2+4r+1)}{8r(x+1)^2((2r-1)x+(r-1))} + \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}}, \qquad r > 1, x \geq 1.$$

When $r \geq 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{9}{10}$. Hence

$$\begin{aligned} g(x) &> \frac{(-2r^2+3r+1)x + (3r^2+4r+1)}{8r(x+1)^2((2r-1)x+(r-1))} + \frac{6x}{5(2x+3)^3} \\ &= \frac{A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5}{40r(x+1)^2(2x+3)^3((2r-1)x+(r-1))} > 0, \end{aligned}$$

where

$$\begin{aligned} &A_1 = 16r^2 + 72r + 40, &&A_2 = 556r + 220, \\ &A_3 = 192r^2 + 1386r + 450, &&A_4 = 588\gamma^2 + 1437r + 405, \\ &A_5 = 13r(3r^2 + 4r + 1) \end{aligned}$$

when $1 < r < 4$, $(\frac{2}{3})^{\frac{1}{r}} > \frac{2}{3}$. Hence

$$\begin{aligned} g(x) &> \frac{-2r^2+3r+1)x + (3r^2+4r+1)}{8r(x+1)^2((2r-1)x+(r-1))} + \frac{8x}{9(2x+3)^3} \\ &= \frac{B_1x^4 + B_2x^3 + B_3x^2 + B_4x + B_5}{72r(x+1)^2(2x+3)^3((2r-1)x+(r-1))} > 0, \end{aligned}$$

where

$$\begin{aligned} &B_1 = -16r^2 + 152r + 72, &&B_2 = -112r^2 + 1068r + 396, \\ &B_3 = 256r^2 + 2562r + 810, &&B_4 = 1036r^2 + 2609r + 729, \\ &B_5 = 243(3r^2 + 4r + 1). \end{aligned}$$

Consequently, we have $\lambda'(x) > g(x) > 0$.

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