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A CLASSIFICATION OF ALL D SUCH THAT $Int(D)$ IS A PRÜFER DOMAIN

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ABSTRACT. Let D be an integral domain with quotient field K. The ring of integer-valued polynomials $Int(D)$ over D is defined by $Int(D) = \{f(x) \in K[x] \mid f(D) \subseteq D\}$. It is known that if $Int(D)$ is a Prüfer domain, then D is an almost Dedekind domain with all residue fields finite. This condition is necessary and sufficient if D is Noetherian, but has been shown to not be sufficient if D is not Noetherian. Several authors have come close to a complete characterization by imposing bounds on orders of residue fields of D and on normalized values of particular elements of D. In this note we give a double-boundedness condition which provides a complete charaterization of all integral domains D such that $Int(D)$ is a Prüfer domain.

1. INTRODUCTION

Let D be an integral domain (not a field) with quotient field K. We define the ring of integer-valued polynomials of D by $Int(D) = \{f(x) \in K[x] | f(D) \subseteq D\}$. Serious study of the ring $Int(D)$ began with papers by Polya [8] and Ostrowski [7], both published in 1919. The main focus of these papers is the structure of $Int(D)$ as a D-module, especially in the case where D is a ring of algebraic integers in a finite degree extension of the field Q of rational numbers.

A problem of more recent interest has been to characterize the integral domains D such that $Int(D)$ is a Prüfer domain. Chabert [2, Corollaire 6.5] and McQuillan [6, Corollary 2.5 and Theorem 5.3] have shown that if D is Noetherian, then $Int(D)$ is a Prüfer domain if and only if D is a Dedekind domain with all residue fields finite.

In the opposite direction, Chabert has shown [2, Proposition 6.3] that if $Int(D)$ is a Prüfer domain, then D must be an almost Dedekind domain with all residue fields finite. (A domain D is almost Dedekind if D_P is a Noetherian valuation domain for all maximal ideals P of D.) However, in contrast to the Noetherian case, Gilmer [4, Example 14] and Chabert [3, Example 6.2] have given (very different) examples of non-Noetherian almost Dedekind domains D with all residue fields finite such that $Int(D)$ is not Prüfer.

In [3] and [4], Chabert and Gilmer both constructed almost Dedekind domains as countably infinite degree algebraic extensions of Dedekind domains. Within that context, Chabert proved that a double-boundedness condition (which is essentially designed to avoid the two negative examples cited) is necessary for $Int(D)$ to be

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Prüfer. He also asked (Q6) if this condition is sufficient. In [5, Theorem 4.1 and Theorem 4.14], Loper translated Chabert's double-boundedness condition to a special class of almost Dedekind domains called sequence domains and demonstrated that in this context, double-boundedness is both necessary and sufficient. More recently, Cahen and Chabert [1, Proposition VI.4.4 and Remark VI.4.5] have essentially extended the sufficiency proof of [5] to cover the general case. However, the necessity of a double-boundedness condition for the general case remains unproven. The necessity proofs that have been given [4, Theorem 13], [3, Theorem 1.2], [5, Theorem 4.1], [1, Proposition VI.4.1], and [9, Theorem 1.9] all involve some type of countability requirement, and with the exception of [5, Theorem 4.1] they all assume a countably generated algebraic field extension. The purpose of this note is to give a simple double-boundedness condition which competely characterizes those non-Noetherian almost Dedekind domains with finite residue fields D for which $Int(D)$ is a Prüfer domain.

2. Double-boundedness

In this section, we give the condition and theorem alluded to in the introduction. First, however, we need to establish one very elementary result and some terminology. We use the symbol • (now and at various other points) to set apart assumptions and terminologies which will hold for the remainder of the text.

- D is a non-Noetherian almost Dedekind domain with all residue fields finite.
- For each maximal ideal P of D, $v_P^{(N)}$ represents the normalized valuation on K associated with P.
- $\text{char}(K) = 0$.

If $\text{char}(K) = 0$, then D contains the ring Z of integers. Then the finiteness of the residue fields of D implies that each maximal ideal P of D must contain a prime number p in Z. To avoid redundancy, we prove our results in the case where $\text{char}(K) = 0$ and then we indicate at the end how the proofs can be modified if K has nonzero characteristic.

For each prime number p which is a nonunit in D, consider the following two sets.

- $F_p = \{|D/P| \mid p \in P\}$
- $E_p = \{v_P^{(N)}(p) \mid p \in P\}$

We are now ready to define the double-boundedness condition alluded to in the introduction.

Definition 2.1. We say that D is *doubly-bounded* provided E_p and F_p are bounded sets for each prime p which is a nonunit in D.

As stated in the introduction, the sufficiency proof involves only minor modifications of the proof given in [1]. The necessity proof requires several preliminary results involving ultrafilters. We begin by introducing more terminology and assumptions.

- $T = \{P_1, P_2, P_3, \cdots\}$ is an infinite sequence of maximal ideals of D such that $\bigcap_{i=1}^{\infty} P_i$ is nonzero.
- U is a nonprincipal ultrafilter on the set T.
- $P_{U,T} = \{d \in D \mid d \in \bigcap_{P_i \in B} P_i \text{ for some set } B \in U\}$.
- $V_{U,T} = \{d \in K \mid v_{P_i}(d) \geq 0 \text{ for all } P_i \in B \text{ for some } B \in U\}$.

Lemma 2.2. *$P_{U,T}$ is a maximal ideal of D.*

The proof of Lemma 2.2 is straightforward and will be omitted.

Lemma 2.3. *$V_{U,T}$ is the valuation domain $D_{P_{U,T}}$.*

Proof. It is easily verified that $V_{U,T}$ is a valuation domain and that it is an overring of D. Then note that $\{d \in K \mid v_{P_i}(d) > 0$ for all $P_i \in B$ for some $B \in U\}$ is the maximal ideal of $V_{U,T}$ and that it contains $P_{U,T}$. □

Now more terminology and assumptions.

- $a \in P_{U,T}$ is a generator of $P_{U,T}D_{P_{U,T}}$.
- $a \in \bigcap_{i=1}^{\infty} P_i$. (Note that there is no generality lost here. If a is not contained in some $P_i \in T$, simply restrict T to those maximal ideals which do contain a and restrict U to this smaller set. $P_{U,T}$ and $V_{U,T}$ will be unchanged.)
- For each $P_i \in T$, v_i is a corresponding valuation on K defined so that $v_i(a) = 1$. (Note that if v_i is normalized, it becomes identical to the valuation $v_{P_i}^{(N)}$ defined previously. Henceforth, we write $v_i^{(N)}$ rather than $v_{P_i}^{(N)}$.)
- v^* is the normalized valuation on K corresponding to $P_{U,T}$.

Lemma 2.4. *For each nonzero element $d \in D$, $v^*(d)$ is the unique nonnegative integer e such that $v_i(d) = e$ for all $P_i \in B$ for some $B \in U$.*

Proof. By definition, we have $v^*(a) = v_i(a) = 1$ for each i. Choose a nonzero element $d \in D$. Let $e = v^*(d)$. Consider the element $\rho = \frac{a^e}{d}$. (Assume that $a^0 = 1$.) Let $C = \{P_i \in T \mid v_i(\rho) > 0\}$ and let $H = \{P_i \in T \mid v_i(\rho) < 0\}$. We know that $C, H \notin U$, since ρ and $\frac{1}{\rho}$ are units in $V_{U,T}$. Hence, $B = \{P_i \in T \mid v_i(\rho) = 0\}$ must lie in U. It follows that $v_i(d) = e$ for all $P_i \in B$. The uniqueness of e is clear. □

We are now prepared for our main result.

Theorem 2.5. *$Int(D)$ is a Prüfer domain if and only if D is doubly-bounded.*

Proof. First assume that D is doubly-bounded. The proof of Proposition VI.4.4 of [1] is sufficient to prove that $Int(D)$ is a Prüfer domain with K_0 taken to be the field Q of rational numbers. (See Remark VI.4.5 in [1] regarding the discrepancy between our hypotheses and those of [1].)

Now assume that D is not doubly-bounded.

Assume that the set F_p is unbounded for some prime p. Also suppose that T is chosen so that p lies in each ideal P_i of T and so that $|D/P_i| < |D/P_{i+1}|$ for each i. Our goal is to show that $Int(D) \subseteq D_{P_{U,T}}[x]$. This will imply that $Int(D)$ is not a Prüfer domain, since $D_{P_{U,T}}[x]$ would then be an overring of $Int(D)$ and not a Prüfer domain. For each i, let $q_i = |D/P_i|$. Choose a nonzero polynomial $h(x) = a_m x^m + \cdots + a_1 x + a_0 \in Int(D)$. For each i define $v_i^{(N)}(h(x)) = \inf_j \{v_i^{(N)}(a_j) \mid 0 \leq j \leq m, a_j \neq 0\}$. Then [3, 1.3 and 1.4] imply that $v_i^{(N)}(h(x)) > \frac{-m}{q_i+1}$ for each i. Hence, if $q_i > m+1$, it follows that $v_i^{(N)}(h(x)) > -1$. Since $q_i > m+1$ for all but finitely many values of i and $v_i^{(N)}$ is normalized, it follows that $v_i^{(N)}(h(x)) \geq 0$ for all but finitely many values of i. Lemma 2.4 then implies that $v^*(a_j) \geq 0$ for each nonzero a_j. Hence, $Int(D) \subseteq D_{P_{U,T}}[x]$, as claimed.

Now suppose that the set E_p is unbounded for some prime p. Also suppose that T is chosen so that p lies in each ideal P_i of T and so that $v_i^{(N)}(p) < v_{i+1}^{(N)}(p)$ for each i. Without loss of generality, we suppose that $v^*(p) = v_i(p)$ for all i. (If not, simply restrict T to the subset which lies in U for which this is true—see Lemma 2.4—and restrict U accordingly.) Recall that $v_i(a) = 1$ for all i. It follows that $v_i^{(N)}(d) = v_i^{(N)}(a)v_i(d)$ for all i and for all nonzero elements d in D. Letting $d = p$ in this identity implies that $v_i^{(N)}(a) \leq v_{i+1}^{(N)}(a)$ for each i. Choose a nonzero polynomial $h(x)$ as in the preceding paragraph and for each i, define $v_i^{(N)}(h(x))$ as above. Again, [3, 1.3 and 1.4] imply that $v_i^{(N)}(h(x)) > \frac{-m}{q_i+1}$ for each i. Hence, $v_i^{(N)}(h(x)) > -m$ for each i. However, we also have

$$v_i^{(N)}(h(x)) = v_i^{(N)}(a)[\inf_j\{v_i(a_j)|0 \leq j \leq m, a_j \neq 0\}.$$

Repeated application of Lemma 2.4 now implies that there exists a set $B \in U$ such that if $P_i \in B$, then $v_i(a_j) = v^*(a_j)$ for $0 \leq j \leq m, a_j \neq 0$. Recall that v^* is normalized. Hence, for $P_i \in B, v_i^{(N)}(h(x))$ is the product of $v_i^{(N)}(a)$ and an integer. It follows that $v_i^{(N)}(h(x))$ either is 0 or is greater than m in absolute value for all but finitely many $P_i \in B$ and hence, without loss of generality, for all $P_i \in B$. Then $v_i^{(N)}(h(x)) > -m$ for each $P_i \in B$ implies that $v_i^{(N)}(h(x)) \geq 0$ for each $P_i \in B$. Then $Int(D) \subseteq D_{P_{U,T}}[x]$ as before. Hence, $Int(D)$ is not a Prüfer domain. This completes the proof. □

If $\text{char}(K) = q$ for some prime number q, then D must contain a finite field F and an element $t \in D$ which is transcendental over F. Moreover, the finiteness of the residue fields of D implies that each maximal ideal P of D must contain an irreducible polynomial $f_P(t) \in F[t]$. The preceding results hold when $\text{char}(K) = q$ with an irreducible polynomial of $F[t]$ replacing the prime number p in the proofs and definitions.

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