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COMPACT QUANTUM GROUPS ASSOCIATED WITH MONOIDAL FUNCTORS

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Abstract. We provide a C^*-algebra structure on the bialgebra associated with a monoidal linear *-functor. The C^*-algebra obtained in this way is a compact quantum group in the sense of Baaj and Skandalis. We show that the category of finite dimensional unitary corepresentations of this C^*-algebra is equivalent to the given category.

Introduction

Woronowicz ([W]) and Baaj and Skandalis ([BS] and [Sk]) defined compact quantum groups as the C^*-algebras generated by the matrix elements of their corepresentations. On the other hand, monoidal categories now form the right framework for the study of quantum groups; see [JS], [JS2], [K], [KT] and [Y]. In [JS], Joyal and Street constructed a bialgebra $\mathrm{End}^\vee(X)$ from a given monoidal functor X satisfying suitable conditions. This bialgebra plays an important role in the modern treatment of Tannaka reconstruction; see [JS], [Sc] and [D].

In this paper we establish the relationship between these two methods. We provide a C^*-algebra structure on $\mathrm{End}^\vee(X)$, and then show that the completion of $\mathrm{End}^\vee(X)$ is a compact quantum group in the Baaj-Skandalis sense. We also show that the category of finite dimensional unitary corepresentations of this C^*-algebra is equivalent to the given C^*-category. This generalizes the result of [W2, Theorem 1.3]. Here concrete monoidal C^*-categories are replaced by abstract monoidal C^*-categories equipped with monoidal linear *-functors.

§1. The bialgebra $\mathrm{End}^\vee(X)$

Throughout this section, $\mathcal{C}$ is a monoidal linear category. We denote by $\mathcal{V}ect_f$ the monoidal category of finite dimensional vector spaces; here the monoidal product is the usual tensor product, and the monoidal unit is the complex numbers $\mathbf{C}$. We assume that $X : \mathcal{C} \longrightarrow \mathcal{V}ect_f$ is a monoidal linear functor.

Recall from [JS, §3] that $\mathrm{End}^\vee(X)$ is defined as the common coequalizer of the maps

$$S \longmapsto (SX(\mu), r), \qquad S \longmapsto (X(\mu)S, s),$$

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from $\mathrm{Hom}(X(s), X(r))$ into $\sum^{\oplus}\{\,\mathrm{End}(X(r)) \,:\, r \in Ob(\mathcal{C})\,\}$, for all $\mu \in \mathrm{Hom}(r,s)$. Thus $\mathrm{End}^{\vee}(X)$ is the quotient of $\sum^{\oplus}\{\,\mathrm{End}(X(r)) \,:\, r \in Ob(\mathcal{C})\,\}$ by the subspace $\mathcal{V}$ generated by elements of the form

$$(SX(\mu), r) \;-\; (X(\mu)S, s),$$

for all $S \in \mathrm{Hom}(X(s), X(r))$ and all $\mu \in \mathrm{Hom}(r,s)$. We write $[S] = (S,r) + \mathcal{V}$ for all $S \in \mathrm{End}(X(r))$.

For each object r of $\mathcal{C}$, we pick a basis $\{\, e_i^r \,\}$ for $X(r)$. Put $e_{i,j}^r = e_i^{r*} \otimes e_j^r$. Then $\{\, e_{i,j}^r \,\}$ is a basis for $\mathrm{End}(X(r))$. We define a linear map $\gamma_r : X(r) \longrightarrow X(r) \otimes \mathrm{End}^{\vee}(X)$ by

$$\gamma_r(u) \;=\; \sum_{i,j} e_{i,j}^r(u) \otimes [e_{j,i}^r], \qquad \forall u \in X(r).$$

Then each γ_r is a comodule structure on $X(r)$, and $\gamma : X \longrightarrow X \otimes \mathrm{End}^{\vee}(X)$ is a monoidal-preserving natural transformation. The vector space $\mathrm{End}^{\vee}(X)$ is a bialgebra with comultiplication Δ_0, counit ϵ_0 and multiplication given by

$$\Delta_0([e_{i,j}^r]) \;=\; \sum_k [e_{i,k}^r] \otimes [e_{k,j}^r],$$
$$\epsilon_0([S]) \;=\; \mathrm{Tr}(S), \qquad [S][T] \;=\; [S \otimes T].$$

Let $\mathcal{C}omod_f(\mathrm{End}^{\vee}(X))$ denote the monoidal category of finite dimensional right $\mathrm{End}^{\vee}(X)$-comodules. Using the natural transformation γ, we obtain a monoidal linear functor

$$\widehat{X} : \mathcal{C} \longrightarrow \mathcal{C}omod_f(\mathrm{End}^{\vee}(X)),$$

given by $\widehat{X}(r) = (X(r), \gamma_r)$. We recall the following standard result from Theorem 3 of [JS, §7].

Theorem 1.1. *If $\mathcal{C}$ is abelian and X is exact and faithful, then $\widehat{X}$ is an equivalence of categories.*

For each object r of $\mathcal{C}$, put

$$\Gamma_r \;=\; \sum_{i,j} e_{i,j}^r \otimes [e_{j,i}^r], \qquad \forall u \in X(r).$$

Then Γ_r is a corepresentation of the bialgebra $\mathrm{End}^{\vee}(X)$, that is,

$$(id \otimes \Delta_0)(\Gamma_r) \;=\; (\Gamma_r \otimes I)(id \otimes \tau)(\Gamma_r \otimes I),$$

where τ is the twist map $\tau(a \otimes b) = b \otimes a$. Since γ is a monoidal-preserving natural transformation, it follows that

$$\Gamma_r \diamond \Gamma_s \;=\; \Gamma_{rs},$$
$$\Gamma_s(X(\mu) \otimes I) \;=\; (X(\mu) \otimes I)\Gamma_r.$$

for all $\mu \in \mathrm{Hom}(r,s)$ and all objects r, s in $\mathcal{C}$. Also note that

$$\gamma_r(u) \;=\; \Gamma_r(u \otimes I), \qquad \forall u \in X(r),$$
$$[S] \;=\; (Tr \otimes id)(\Gamma_r(S \otimes I)), \qquad \forall S \in \mathrm{End}(X(r)).$$

Suppose that $\mathcal{C}$ is left autonomous. We recall from [JS2, §7] that a monoidal category $\mathcal{C}$ is said to be left autonomous if for any object r, there is an object $\tilde{r}$ and there are arrows $\vartheta_r \in \mathrm{Hom}(\iota, r\tilde{r})$ and $\bar{\vartheta}_r \in \mathrm{Hom}(\tilde{r}r, \iota)$ such that

$$\begin{aligned}(I_r \diamond \bar{\vartheta}_r)(\vartheta_r \diamond I_r) &= I_r,\\ (\bar{\vartheta}_r \diamond I_{\tilde{r}})(I_{\tilde{r}} \diamond \vartheta_r) &= I_{\tilde{r}},\end{aligned}$$

where $\diamond$ is the monoidal product, and ι is the monoidal unit of $\mathcal{C}$. We call $\tilde{r}$ a left dual of r, and we refer to the pair $(\vartheta_r, \bar{\vartheta}_r)$ as an adjunction between $\tilde{r}$ and r. We then deduce that that there is a bijective antilinear map $J_r : X(r) \longrightarrow X(\tilde{r})$ such that

$$\begin{aligned}X(\vartheta_r)(1) &= \sum_i e_i^r \otimes J_r e_i^r,\\ X(\bar{\vartheta}_r)(J_r u_2 \otimes u_1) &= \langle u_1, u_2^* \rangle, \qquad \forall u_1, u_2 \in X(r),\end{aligned}$$

where u_2^* is the image of u_2 in the dual space $X(r)^*$. For each $S \in \mathrm{End}(X(r))$, put

$$S^\sharp = (X(\bar{\vartheta}_r) \otimes I_{\tilde{r}})(I_{\tilde{r}} \otimes S \otimes I_{\tilde{r}})(I_{\tilde{r}} \otimes X(\vartheta_r)).$$

Then $\mathrm{End}^\vee(X)$ admits an antipode ν (see [JS, §9]) given by $\nu([S]) = [S^\sharp]$.

§2. Compact quantum groups associated with monoidal functors

Throughout this section, $\mathcal{R}$ is a strict monoidal abelian C^*-category; we assume that the monoidal unit ι is irreducible. Note that since $\mathcal{R}$ is abelian, it follows that $\mathcal{R}$ has subobjects and direct sums in the sense of [DR, §1]. We denote by $\mathcal{H}il_f$ the strict monoidal C^*-category of finite dimensional Hilbert spaces; here the monoidal product is the usual tensor product, and the monoidal unit is the complex numbers $\mathbf{C}$. If Q is a compact quantum group, we denote by $\mathcal{U}corep_f(Q)$ the strict monoidal C^*-category of finite dimensional unitary corepresentations of Q.

Observe that if $\mathcal{R}$ is left autonomous, then it is also right autonomous. If $(\vartheta_r, \bar{\vartheta}_r)$ is an adjunction between $\tilde{r}$ and r, then $(\bar{\vartheta}_r^*, \vartheta_r^*)$ is an adjunction between r and $\tilde{r}$. We will choose $\vartheta_{\tilde{r}} = \bar{\vartheta}_r^*$ and $\bar{\vartheta}_{\tilde{r}} = \vartheta_r^*$.

The category $\mathcal{H}il_f$ is left autonomous in the following way. For each object V of $\mathcal{H}il_f$, let W be an object of $\mathcal{H}il_f$ equipped with a bijective antilinear map $J : V \longrightarrow W$. Pick an orthonormal basis $\{e_i\}$ for V, and define

$$\begin{aligned}t_J(1) &= \sum_i e_i \otimes Je_i,\\ \bar{t}_J(Jv_2 \otimes v_1) &= \langle v_1 | v_2 \rangle, \qquad \forall v_1, v_2 \in V.\end{aligned}$$

Then W is a left dual of V with an adjunction $(t_J, \bar{t}_J)$. Also V is a left dual of W with an adjunction $(\bar{t}_J^*, t_J^*)$. Note that $\bar{t}_J^* = t_{J^{-1}}$ and $t_J^* = \bar{t}_{J^{-1}}$, with respect to an orthonormal basis $\{f_i\}$ of W.

If Q is a compact quantum group, then the category $\mathcal{U}corep_f(Q)$ is left autonomous in the following way. For each object (α, V_α) of $\mathcal{U}corep_f(Q)$, put $\widetilde{\alpha} = \alpha^{J\otimes *}$, where $J : V_\alpha \longrightarrow \tilde{V}_\alpha$ is the canonical antilinear map. Since α is unitary, it follows that $t_J \in \mathrm{Hom}(\iota, \alpha \diamond \widetilde{\alpha})$ and $\bar{t}_J \in \mathrm{Hom}(\widetilde{\alpha} \diamond \alpha, \iota)$. Thus $(\widetilde{\alpha}, \tilde{V}_\alpha)$ is a left dual of (α, V_α) with an adjunction $(t_J, \bar{t}_J)$.

Theorem 2.1. *Suppose that $\mathcal{R}$ is a left autonomous strict monoidal abelian C^*-category. Assume that there is an exact faithful monoidal linear *-functor X :*

$\mathcal{R} \longrightarrow \mathcal{H}il_f$. *Then there is a* C^**-norm on* $\mathrm{End}^\vee(X)$*, and the completion* Q *of* $\mathrm{End}^\vee(X)$ *under this norm is a compact quantum group.*

Let r be any object of $\mathcal{R}$. Since $X : \mathcal{R} \longrightarrow \mathcal{H}il_f$ is faithful, $\mathrm{Hom}(r,r)$ is a finite dimensional C^*-algebra. Hence there are minimal projections $\phi_i \in \mathrm{Hom}(r,r)$ such that $\phi_i\phi_j = \delta_{ij}$ and $\sum_i \phi_i = I_r$. Since $\mathcal{R}$ has subobjects, there are arrows $\mu_i \in \mathrm{Hom}(r_i, r)$ and $\mu_i^* \in \mathrm{Hom}(r, r_i)$ such that $\mu_i\mu_i^* = \phi_i$ and $\mu_i^*\mu_i = I_{r_i}$. Since the ϕ_i are minimal, the r_i are irreducible. Since $\phi_i\phi_j = \delta_{ij}$, we have $\mu_i^*\mu_j = \delta_{ij}$. Therefore $r = \sum_i^\oplus r_i$.

Put $t_r = X(\vartheta_r)$ and $\bar{t}_r = X(\bar{\vartheta}_r)$. Since X is monoidal, $(t_r, \bar{t}_r)$ is an adjunction between $X(\tilde{r})$ and $X(r)$. Let $\tilde{J} : X(r) \longrightarrow \widetilde{X(r)}$ be the canonical antilinear map. Then $(\bar{t}_r \otimes I)(I \otimes t_{\tilde{J}})$ is a bijective linear map from $X(\tilde{r})$ onto $\widetilde{X(r)}$, and hence $\dim(X(\tilde{r})) = \dim(\widetilde{X(r)})$. Therefore there is a bijective antilinear map $J : X(r) \longrightarrow X(\tilde{r})$ such that $\langle Ju_1|Ju_2\rangle = \langle u_2|u_1\rangle$ for all $u_1, u_2 \in X(r)$. Put $\psi = (\bar{t}_r \otimes I)(I \otimes t_J)$; then it is a bijective linear map on $X(\tilde{r})$. We then deduce that

$$t_r \;=\; (I \otimes \psi^{-1})t_J, \qquad \bar{t}_r \;=\; \bar{t}_J(\psi \otimes I).$$

Put $J_r = \psi^{-1}J$. Then we have

$$t_r \;=\; t_{J_r}, \qquad \bar{t}_r \;=\; \bar{t}_{J_r}.$$

Put $t_{\tilde{r}} = \bar{t}_r^*$ and $\bar{t}_{\tilde{r}} = t_r^*$. Since X is a *-functor, we get

$$t_{\tilde{r}} \;=\; X(\vartheta_{\tilde{r}}), \qquad \bar{t}_{\tilde{r}} \;=\; X(\bar{\vartheta}_{\tilde{r}}).$$

For each object r of $\mathcal{R}$, we pick an orthonormal basis $\{e_i^r\}$ for $X(r)$.

Proposition 2.2. *For any* $r_1, \ldots, r_n \in \mathcal{J}(\mathcal{R})$*, the set*

$$\{\ [e_{ij}^{r_k}]\ :\ i,j = 1, \ldots, \dim(X(r_k)),\ k = 1, \ldots, n\ \}$$

is linearly independent.

Proof. Let W be the vector space generated by the set

$$\{\ [e_{ij}^{r_k}]\ :\ i,j = 1, \ldots, \dim(X(r_k)),\ k = 1, \ldots, n\ \}.$$

To prove this proposition, it is sufficient to show that for any finite set of complex numbers

$$\{\ c_{ij}^k\ :\ i,j = 1, \ldots, \dim(X(r_k)),\ k = 1, \ldots, n\ \},$$

there is a linear functional $\rho \in W^*$ such that $\rho([e_{ij}^{r_k}]) = c_{ij}^k$ for all i, j and k. Let $B = \sum_k^\oplus \mathrm{End}(X(r_k))$, and put

$$A \;= \{\sum_k{}^{\oplus}(id \otimes \rho)(\Gamma_{r_k})\ :\ \rho \in W^*\ \}.$$

Then A is a subalgebra of B. Since the functor $\widehat{X}$ of Theorem 1.1 is fully faithful, it follows that $\Gamma_{r_1}, \ldots, \Gamma_{r_n}$ are pairwise nonequivalent irreducible. We then deduce that $X(r_1), \ldots, X(r_n)$ are pairwise nonisomorphic simple A-modules. Therefore

$$A' \;=\; \mathrm{End}_A\Big(\sum_k{}^{\oplus} X(r_k)\Big) \;=\; \sum_k{}^{\oplus}\mathbf{C}I_{r_k}.$$

Thus B is contained in the bicommutant A''. By the Jacobson density theorem, for any $T = \sum_k^\oplus T_k \in B$, there is an element $a = \sum_k^\oplus (id \otimes \rho)(\Gamma_{r_k}) \in A$ such that

$ae_i^{r_k} = Te_i^{r_k}$ for all i and k. For each $k = 1, \dots, n$, if we take $T_k = \sum_{ij} c_{ji}^k e_{ij}^{r_k}$, then we get

$$\sum_{ij} \rho([e_{ji}^{r_k}]) e_{ij}^{r_k} \ = \ \sum_{ij} c_{ji}^k e_{ij}^{r_k}. \quad \blacksquare$$

Proposition 2.3. *Put* $Q_0 = \sum_{r \in \mathcal{J}(\mathcal{R})}^{\oplus} [\mathrm{End}(X(r))]$. *Then* $\mathrm{End}^\vee(X) = Q_0$.

Proof. Let $S \in \mathrm{End}(X(s))$. Choose a decomposition $I_s = \sum_i \mu_i \mu_i^*$ with $\mu_i \in \mathrm{Hom}(s_i, s)$ and $s_i \in \mathcal{J}(\mathcal{R})$. Then we have

$$[S] \ = \ \sum_i [X(\mu_i^*) S X(\mu_i)].$$

Therefore $\mathrm{End}^\vee(X)$ is generated by

$$\{ \ [R] \ : \ R \in \mathrm{End}(X(r)), \ r \in \mathcal{J}(\mathcal{R}) \ \}.$$

We deduce from Proposition 2.2 that

$$\{ \ [e_{ij}^r] \ : \ i,j = 1, \dots, \dim(X(r)), \ r \in \mathcal{J}(\mathcal{R}) \ \}$$

is a basis for $\mathrm{End}^\vee(X)$, and this proves the proposition. $\blacksquare$

We define an involution operation * on $\mathrm{End}^\vee(X)$ by

$$[S]^* \ = \ [S^{*\sharp}] \ = \ [\tilde{S}].$$

Then $\mathrm{End}^\vee(X)$ becomes a unital *-algebra.

For each object r of $\mathcal{R}$, we have

$$\widetilde{\Gamma}_r \ = \ \sum_{ij} \widetilde{e_{ij}^r} \otimes [\widetilde{e_{ji}^r}] \ = \ \sum_{ij} e_{ij}^{\tilde{r}} \otimes [e_{ji}^{\tilde{r}}] \ = \ \Gamma_{\tilde{r}}.$$

Since $\vartheta_r \in \mathrm{Hom}(\iota, r\tilde{r})$ and $\bar{\vartheta}_r \in \mathrm{Hom}(\tilde{r}r, \iota)$, it follows that

$$(\Gamma_r \diamond \widetilde{\Gamma}_r)(t_r \otimes I) \ = \ t_r \otimes I,$$
$$(\bar{t}_r \otimes I)(\widetilde{\Gamma}_r \diamond \Gamma_r) \ = \ \bar{t}_r \otimes I.$$

Therefore Γ_r is a unitary.

We define a linear functional h_0 on Q_0 by

$$h_0([S_r]) \ = \ \begin{cases} 1, & \text{if } [S_r] = [1]; \\ \\ 0, & \text{if } S_r \in \mathrm{End}(X(r)), \ r \neq \iota. \end{cases}$$

Proposition 2.4. *Let* $r, s \in \mathcal{J}(\mathcal{R})$ *and* $r \neq s$.

(i) *For any* $S_r \in \mathrm{End}(X(r))$ *and* $S_s \in \mathrm{End}(X(s))$,

$$h_0([S_s]^*[S_r]) \ = \ 0.$$

(ii) *For any* $S_r, \ T_r \in \mathrm{End}(X(r))$,

$$h_0([T_r]^*[S_r]) \ = \ \dim(X(r))^{-1} \sum_m \langle S_r e_m^r | T_r e_m^r \rangle.$$

(iii) *For any nonzero* $a = \sum_r^{\oplus} [S_r] \in Q_0$,

$$h_0(a^*a) \ = \ \sum_r \dim(X(r))^{-1} \sum_m \|S_r e_m^r\|^2 \ > \ 0.$$

Proof. (i) Choose a decomposition $I_{\tilde{r}s} = \sum_i \mu_i \mu_i^*$ with $\mu_i \in \mathrm{Hom}(r_i, \tilde{r}s)$ and $r_i \in \mathcal{J}(\mathcal{R})$. We then have

$$\begin{aligned} h_0([S_s]^*[S_r]) &= h_0([\tilde{S}_s \otimes S_r]) \\ &= \sum_{r_i=\iota} h_0([X(\mu_i^*)(\tilde{S}_s \otimes S_r)X(\mu_i)]). \end{aligned}$$

Note that for any objects r, s and t, $\mu \longmapsto (I_{\tilde{r}} \diamond \mu)(\vartheta_{\tilde{r}} \diamond I_t)$ is a bijective linear map from $\mathrm{Hom}(rt, s)$ onto $\mathrm{Hom}(t, \tilde{r}s)$. Since $\mathrm{Hom}(r, s) = \{0\}$, it follows that $X(\mu_i) = 0$ for all i with $r_i = \iota$. This proves (i).

(ii) By similar arguments as in (i) with $r = s$, we get

$$h_0([T_r]^*[S_r]) = \sum_{r_i=\iota} h_0([X(\mu_i^*)(\tilde{T}_s \otimes S_r)X(\mu_i)]).$$

Since $\mathrm{Hom}(\iota, \tilde{r}r) = \mathbf{C}\bar{\vartheta}_r^*$ and $\mathrm{Hom}(\tilde{r}r, \iota) = \mathbf{C}\bar{\vartheta}_r$, it follows that

$$\sum_{r_i=\iota} X(\mu_i)X(\mu_i^*) = c\, X(\bar{\vartheta}_r^*)X(\bar{\vartheta}_r),$$

for some scalar c. For any i with $r_i \neq \iota$, we have $\mu_i^* \bar{\vartheta}_r^* \in \mathrm{Hom}(\iota, r_i) = \{0\}$. Hence

$$\begin{aligned} \bar{t}_r^* &= X(\bar{\vartheta}_r^*) = X\left(\sum_i \mu_i \mu_i^* \bar{\vartheta}_r^*\right) = X\left(\sum_{r_i=\iota} \mu_i \mu_i^* \bar{\vartheta}_r^*\right) \\ &= c\, X(\bar{\vartheta}_r^*)X(\bar{\vartheta}_r)X(\bar{\vartheta}_r^*) = c\, \bar{t}_r^* \bar{t}_r \bar{t}_r^*. \end{aligned}$$

Thus $c\, \bar{t}_r \bar{t}_r^*(1) = 1$. Observe that

$$\begin{aligned} \bar{t}_r \bar{t}_{\tilde{r}}^*(1) &= \sum_i \langle J_r^{-1} e_i^{\tilde{r}} | J_r^{-1} e_i^{\tilde{r}} \rangle = \dim(X(r)), \\ \bar{t}_r(\tilde{T}_r \otimes S_r)\bar{t}_r^*(1) &= \sum_m \langle S_r e_m^r | T_r e_m^r \rangle. \end{aligned}$$

Now we have

$$\begin{aligned} h_0([T_r]^*[S_r]) &= h_0\left([\sum_{r_i=\iota} X(\mu_i)X(\mu_i^*)(\tilde{T}_s \otimes S_r)\ ]\right) \\ &= c\, h_0\left([X(\bar{\vartheta}_r^*)X(\bar{\vartheta}_r)(\tilde{T}_s \otimes S_r)]\right) \\ &= c\, h_0\left([\bar{t}_r(\tilde{T}_s \otimes S_r)\bar{t}_r^*]\right) \\ &= c\, h_0\left([\sum_m \langle S_r e_m^r | T_r e_m^r \rangle 1\ ]\right) \\ &= \dim(X(r))^{-1} \sum_m \langle S_r e_m^r | T_r e_m^r \rangle. \end{aligned}$$

(iii) It is a consequence of (i) and (ii). ■

Proof of Theorem 2.1. For each $a \in Q_0$, we put

$$\|a\| = \sup\{\|\pi(a)\| : \pi \text{ is a nondegenerate representation on Hilbert spaces}\}.$$

Since each Γ_r is unitary, $\|a\|$ is finite and hence $\|\cdot\|$ is a C^*-seminorm on Q_0. The ideal $\mathcal{I}_0$ of Q_0 consisting of elements of seminorm zero is closed under the involution operation *. The canonical quotient map $q : Q_0 \longrightarrow Q_0/\mathcal{I}_0$ is a unital *-homomorphism. Let Q denote the completion of $Q_0/\mathcal{I}_0$. We represent the C^*-algebra Q on a Hilbert space by a faithful nondegenerate representation, and then see that

$$\|(q \otimes q) \circ \Delta_0(a)\| \;\leq\; \|a\|, \qquad \forall a \in Q_0.$$

We then deduce that there is a comultiplication $\Delta : Q \longrightarrow Q \otimes Q$ such that

$$\Delta(q(a)) \;=\; (q \otimes q) \circ \Delta_0(a), \qquad \forall a \in Q_0.$$

Each $\beta_r = (id \otimes q)(\Gamma_r)$ is a unitary corepresentation of (Q, Δ), and the matrix elements of the family $\{\beta_r\}$ generate Q. By Remark 2.6(b) of [B], there exists a unique Haar measure h for $(Q, \Delta, \{\beta_r\})$. Let $r \in \mathcal{J}(\mathcal{R})$ with $r \neq \iota$. Since the unit I of Q_0 is not in $\mathcal{I}_0$, it follows that $q(I)$ is not in the linear span of $q([e^r_{ij}])$ for all i, j. By the Hahn-Banach theorem, we can choose $\eta \in Q^*$ such that $\eta(q(I)) = 1$ and $\eta(q([e^r_{ij}])) = 0$ for all i, j. Using Theorem 2.3 of [B], we get

$$\begin{aligned} h(q([e^r_{ij}])) \;&=\; (\eta * h)(q([e^r_{ij}])) \\ &=\; \sum_k \eta(q([e^r_{ik}]))h(q([e^r_{kj}])) \;=\; 0. \end{aligned}$$

Therefore $h \circ q = h_0$. If π is the cyclic representation of Q induced by the positive linear functional h, then it follows Proposition 2.4(iii) that $\pi \circ q$ is a faithful nondegenerate representation of Q_0. Therefore $\mathcal{I}_0 = \{0\}$ and $\|\cdot\|$ is a C^*-norm. ■

The following theorem contains Theorem 1.3 of [W2] as a special case.

Theorem 2.5. *The functor* $\Pi : \mathcal{R} \longrightarrow \mathcal{U}corep_f(Q)$*, given by* $\Pi(r) = \Gamma_r$ *and* $\Pi(\mu) = X(\mu)$*, is an equivalence of monoidal categories.*

Proof. Since $\widehat{X}$ is fully faithful, so is Π. To prove Π is an equivalence, let α be any object of $\mathcal{U}corep_f(Q)$. We need to prove that α is equivalent to $\Pi(r)$ for some r in $\mathcal{R}$. By Theorem 2.4 of [B], α is the direct sum of irreducible unitary subcorepresentations α_i. The family $\mathcal{L} = \{\, \Gamma_r \,:\, r \in \mathcal{J}(\mathcal{R}) \,\}$ consists of mutually nonequivalent irreducible corepresentations of Q, and the matrix elements of $\mathcal{L}$ generate Q_0. By Theorem 2.5 of [B], each α_i is equivalent to an element Γ_{r_i} of $\mathcal{L}$. If we put $r = \sum^{\oplus} r_i$, then $\Pi(r) = \Gamma_r$ is equivalent to $\sum^{\oplus} \Gamma_{r_i}$. Hence α is equivalent to $\Pi(r)$. ■

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