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HARDY'S THEOREM FOR THE n-DIMENSIONAL EUCLIDEAN MOTION GROUP

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Abstract. An uncertainty principle, due to Hardy, for Fourier transform pairs on $\mathbb{R}$ says that if the function f is "very rapidly decreasing", then the Fourier transform cannot also be "very rapidly decreasing" unless f is identically zero. In this paper we state and prove an analogue of Hardy's theorem for the n-dimensional Euclidean motion group.

1. Introduction

It is a well-known simple fact that if a function f on $\mathbb{R}$ is compactly supported then its Fourier transform $\tilde{f}$ cannot also be compactly supported, unless $f = 0$. More generally, we have the following principle in classical Fourier analysis : If the function f is "very rapidly decreasing" then the Fourier transform cannot also be "very rapidly decreasing", unless f is identically zero. The following result of Hardy makes the rather vague statement above precise:

Theorem 1.1 (Hardy). *Suppose f is a measurable function on $\mathbb{R}$ such that*

$$| f(x) | \leq Ce^{-\alpha x^2}, \ | \tilde{f}(\xi) | \leq Ce^{-\beta \xi^2}, \ x, \xi \in \mathbb{R}, \tag{1.1}$$

where α, β and C are positive constants. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying (1.1), and if $\alpha\beta = \frac{1}{4}$ then $f(x) = Ce^{-\alpha x^2}$.

For a proof of the above theorem see [2], Theorem 3.2. Hardy's theorem is also valid in $\mathbb{R}^n$ (see [8] for a proof). A generalization of Hardy's theorem, due to Cowling and Price, asserts that if a, b are nonnegative constants such that $ab \geq \frac{1}{4}$, then the only $f \in \mathcal{S}'$ satisfying $\|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty$ for $1 \leq p, q \leq \infty$ with at least one of them finite, is $f = 0$. On the other hand, if $ab < \frac{1}{4}$, there are infinitely many $f \in \mathcal{S}$ satisfying $\|e^{ax^2} f\|_p + \|e^{by^2} \hat{f}\|_q < \infty$ (see [1]). Another theorem of this kind is due to A. Beurling [5], which says that if $f \in L^1(\mathbb{R})$ is such that $\int\int_{\mathbb{R}^2} | f(x)\hat{f}(y) | e^{|x\, y|} dx\, dy < \infty$, then $f = 0$ a.e. One can see that Hardy's theorem can be deduced from this more general theorem of Beurling. This

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class of results can also be viewed as some sort of "uncertainty principle". For an elaboration of this point of view, see [6], [8] and the bibliographies in those papers.

Suppose G is a "sufficiently nice" connected Lie group with Haar measure m, and $\widehat{G}$ its unitary dual. Then by the abstract Plancherel theorem we know that there exist a measure structure and a unique measure μ on $\widehat{G}$ such that for all $f \in L^1(G) \cap L^2(G)$,

$$\int_G \mid f(x) \mid^2 dm(x) \ = \ \int_{\widehat{G}} tr(\pi(f)\pi(f)^*) d\mu(\pi),$$

where for f in $L^1(G)$ we define the group Fourier transform $\hat{f}$ of f by

$$\hat{f}(\pi) = \pi(f) \ = \ \int_G f(x)\pi(x) dm(x), \ \pi \in \widehat{G}$$

(the integral being interpreted suitably). Therefore we can ask the following question in this more general set up : Suppose f is an L^1-function on G such that both f and $\hat{f}$ decay "very rapidly" at infinity. Then is $f \ = \ 0$ a.e.?

Analogues of Hardy's theorem for the Heisenberg group $\mathcal{H}_n$ and the Euclidean motion group of the plane $M(2)$, have been proved in [8]. In the next two sections we shall state and prove an analogue of Hardy's theorem for the n-dimensional Euclidean motion group, $M(n)$, $n \geq 2$. While the proof in [8] for $M(2)$ proceeds by reducing the theorem to the Euclidean case, the proof here is more direct and involves some simple estimates of the K-finite matrix coefficients of irreducible representations.

Finally, we remark that, in [7], an analogue of Hardy's theorem is proved for a subclass of connected noncompact semi-simple Lie groups and all symmetric spaces of the noncompact type.

2. Description of the unitary dual of $M(n)$

The group $G \ = \ M(n)$ is the semi-direct product of $\mathbb{R}^n$ with the special orthogonal group $K \ = \ SO(n)$. A typical element of G is denoted by (a, k) where $a \in \mathbb{R}^n$ and $k \in K$. If da denotes Lebesgue measure on $\mathbb{R}^n$ and dk normalized Haar measure on K, then Haar measure on G is given by $da\ dk$. The natural action of K on $\mathbb{R}^n$ is denoted by $k \cdot \nu$, where $k \in K$ and $\nu \in \mathbb{R}^n$. (Since the 'natural' action is left multiplication by the matrix k, $\mathbb{R}^n$ should really be thought of as the space of column vectors.) For any unexplained terminology and notation in this section the reader may refer to [4].

We shall now describe $\widehat{G}$, the unitary dual of G.

Let $\nu \in \mathbb{R}^n$ and $\nu \neq 0$. Let U_ν denote the stabilizer of ν in K under the natural action of K on $\mathbb{R}^n$. Then U_ν is conjugate to the subgroup $\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \ : \ A \in SO(n-1)\}$. We identify this subgroup with $SO(n-1)$. Fix an irreducible unitary representation λ of U_ν acting on $\mathbb{C}^{d_\lambda}$. Let $H(K, \lambda)$ be the vector space of all measurable functions $\psi : K \to \mathbb{C}^{d_\lambda}$ such that $\psi(uk) \ = \lambda(u)(\psi(k))$ for $u \in U_\nu$, $k \in K$ and $\int_K \|\psi(k)\|^2 dk < \infty$. Here $\| \cdot \|$ denotes the norm on $\mathbb{C}^{d_\lambda}$. It is easy to see that $H(K, \lambda)$ is a Hilbert space with respect to the inner product defined by

$$(\psi_1, \psi_2) \ = \ d_\lambda \int_K \langle \psi_1(k), \psi_2(k) \rangle dk$$

where $\langle .,. \rangle$ denotes the usual inner product on $\mathbb{C}^{d_\lambda}$ and $\psi_1,\ \psi_2 \in H(K,\lambda)$. Define $T_{\nu,\lambda}$ on $H(K,\lambda)$ by

$$(T_{\nu,\lambda}(a,k)\psi)(k_o) \ = \ e^{i\langle k_o^{-1}\cdot\nu, a\rangle}\ \psi(k_o k),\ \psi \in H(K,\lambda) \tag{2.1}$$

for $a \in \mathbb{R}^n$, $k, k_o \in K$. We also use $\langle .,. \rangle$ to denote the inner product on $\mathbb{R}^n$. One can easily verify that $T_{\nu,\lambda}$ is a unitary representation of G on $H(K,\lambda)$. Further, it can be shown that (see [3], [4]):

(a) For $\nu \neq 0$ and any $\lambda \in \widehat{U_\nu}$, the representation $T_{\nu,\lambda}$ is irreducible.

(b) Every infinite dimensional irreducible unitary representation of G is equivalent to some $T_{\nu,\lambda}$ with ν and λ as above.

(c) Given two non-zero vectors $\nu,\ \nu_1 \in \mathbb{R}^n$ and representations $\lambda \in \widehat{U_\nu}$ and $\lambda_1 \in \widehat{U_{\nu_1}}$, the representations $T_{\nu,\lambda}$ and T_{ν_1,λ_1} are equivalent if and only if ν and ν_1 belong to the same K-orbit (i.e. $\nu,\ \nu_1$ have the same Euclidean norm) and the representations λ and λ_1 are equivalent under the obvious identification of U_ν with U_{ν_1}.

If $\|\nu\| = \|\nu_1\| = r,\ r \in \mathbb{R}^+$, then by abuse of notation we denote the n-tuple $(0, 0, \cdots, 0, r)^t$ also by r. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$ and t denotes the transpose. In this case we write U_r for U_ν and note that U_r consists precisely of the matrices $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with $A \in SO(n-1)$. Hence, we adopt the notation $U_r \ = \ SO(n-1)$. We then choose the representative of the equivalence class of $T_{\nu,\lambda}$ as $T_{r,\lambda}$. Apart from these infinite dimensional representations $T_{r,\lambda}$, the finite dimensional unitary representations of K also yield finite dimensional unitary representations of G, but these do not enter into the Plancherel formula (see [4] for details).

The Plancherel measure μ is supported on the subset of $\widehat{G}$ given by $\{T_{r,\lambda}\ :\ \lambda \in SO(n-1)\hat{}$ and $r \in \mathbb{R}^+\}$ and on each "piece" $\{T_{r,\lambda}\ :\ r \in \mathbb{R}^+\}$ with $\lambda \in SO(n-1)\hat{}$ fixed, it is given by $C_n r^{n-1} dr$, where C_n is a constant depending only on n.

Before we end this section we state the following lemma, from complex analysis, that plays a crucial role in the proof of our main theorem:

Lemma 2.1. *Suppose h is an entire function on $\mathbb{C}$ such that $h(z) \ = O(e^{a|z|^2})$ for $z \in \mathbb{C}$ and $h(t) \ = \ O(e^{-at^2})$ for $t \in \mathbb{R}$ where a is a positive constant. Then $h(z) \ = \ Const.e^{-az^2}$, $z \in \mathbb{C}$.*

Applying the following result (see [10], pp.175) to the even and odd parts of the function separately, the lemma follows: Let h be an entire function on $\mathbb{C}$ such that $h(z) \ = \ O(e^{a|z|})$ for $z \in \mathbb{C}$ and $h(t) \ = \ O(e^{-at})$ for $t \in \mathbb{R}^+$, where 'a' is a positive constant. Then $h(z) \ = \ Const.e^{-az}$, $z \in \mathbb{C}$.

3. ANALOGUE OF HARDY'S THEOREM FOR $M(n)$

We will now define the group Fourier transform on $G \ = \ M(n)$. Given a function f in $L^1(G)$ and $\pi \in \widehat{G}$ the group Fourier transform $\hat{f}$ of f at π is the operator

$$\hat{f}(\pi) \ = \ \pi(f) \ = \ \int_{\mathbb{R}^n}\int_K f(a,k)\pi(a,k)dkda \tag{3.1}$$

(the integral being interpreted suitably, see [9]). Then by the Plancherel theorem we know that for $f \in L^1 \cap L^2(G)$, $\hat{f}$ is a Hilbert-Schimdt operator for almost all π (with respect to the Plancherel measure) and we denote its Hilbert-Schimdt norm by $\|\hat{f}(\pi)\|_{HS}$. We now state and prove an analogue of Hardy's theorem for G.

Theorem 3.1. *Suppose f is a measurable function on G satisfying the following estimates:*

$$| f(a,k) | \leq \; Ce^{-\alpha\|a\|^2}, \; (a,k) \in G, \tag{3.2}$$

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \; \leq \; C_\lambda e^{-\beta r^2}, \; r \in \mathbb{R}^+, \tag{3.3}$$

for some positive constants C_λ, α, β and C where C_λ depends only on λ. If $\alpha\beta \; > \; \frac{1}{4}$ then $f \; = \; 0$ a.e.

(*Remark* 3.2. Since functions on $\mathbb{R}^n$ can be thought of as functions on G invariant under right action by K, Hardy's theorem for $\mathbb{R}^n$ shows that $\frac{1}{4}$ is the best possible constant.)

Proof. Observe that by identifying $-r$ with the n-tuple $(0, \cdots, 0, -r)^t$ for $r \in \mathbb{R}^+$ we can define $T_{-r,\lambda}$. Now, $T_{-r,\lambda}$ and $T_{r,\lambda}$ are equivalent as representations of G. Hence $\|\hat{f}(T_{-r,\lambda})\|_{HS} = \|\hat{f}(T_{r,\lambda})\|_{HS}$ and we thus have

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \; \leq \; C_\lambda e^{-\beta r^2}, \; r \in \mathbb{R}. \tag{3.4}$$

For $r \in \mathbb{R}$ and $\lambda \in SO(n-1)\widehat{\ }$, let $S \; = \; \{ \; e_i^\lambda \; : \; i \in \mathbb{N} \; \}$ be a basis of $H(K,\lambda)$ consisting of K-finite vectors. (For fixed λ, notice that the representation $T_{r,\lambda}$ restricted to K is just the right regular action of K on $H(K,\lambda)$.) Note that if ϕ is a K-finite vector, then $\phi \in C^\infty(K, \mathbb{C}^{d_\lambda})$. It suffices to show that for any fixed i and j, the condition $\alpha\beta \; > \; \frac{1}{4}$ implies $(\hat{f}(T_{r,\lambda})e_i^\lambda, e_j^\lambda) \equiv 0$ as a function of r and λ. Fix $i_o, j_o \in \mathbb{N}$ and consider for $r \in \mathbb{R}$,

$$(\hat{f}(T_{r,\lambda})e_{i_o}^\lambda, e_{j_o}^\lambda) \; = \; \int_K \int_{\mathbb{R}^n} f(a,k)(T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda) da dk. \tag{3.5}$$

Let $\Phi_{r,\lambda}^{i_o,j_o}(a,k) \; = \; (T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda)$ for $r \in \mathbb{R}$, $\lambda \in SO(n-1)\widehat{\ }$, $i_o, j_o \in \mathbb{N}$, and $(a,k) \in G$. Then by definition of $T_{r,\lambda}$, we have

$$\begin{aligned} \Phi_{r,\lambda}^{i_o,j_o}(a,k) &= d_\lambda \int_K \langle (T_{r,\lambda}(a,k)e_{i_o}^\lambda)(k_o), e_{j_o}^\lambda(k_o)\rangle dk_o \\ &= d_\lambda \int_K e^{i\langle k_o^{-1}\cdot r, a\rangle} \langle e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o)\rangle dk_o \\ &= d_\lambda \int_K e^{i\langle r, k_o\cdot a\rangle} \langle e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o)\rangle dk_o. \end{aligned} \tag{3.6}$$

Here the real number r is identified with $(0, \cdots, 0, r)^t$ and $\langle ., . \rangle$ denotes both inner product on $\mathbb{R}^n$ as well as $\mathbb{C}^{d_\lambda}$. Notice that the integral on the right-hand side makes sense even when $r \in \mathbb{C}$ where we identify $r \in \mathbb{C}$ with $(0, \cdots, 0, r)^t$ in $\mathbb{C}^n$ and $\langle ., . \rangle$ now denotes inner product on $\mathbb{C}^n$ also. Hence, with (a,k) fixed, the function $\Phi_{r,\lambda}^{i_o,j_o}(a,k)$ of the variable r extends to the whole complex plane. One can easily see that for fixed (a,k), $z \mapsto \Phi_{z,\lambda}^{i_o,j_o}(a,k)$ is an entire function on $\mathbb{C}$. Moreover, for

$z \in \mathbb{C}$,

$$
\begin{aligned}
| \Phi^{i_o,j_o}_{z,\lambda}(a,k) | \ &\le\ d_\lambda \int_K | e^{i\langle z, k_o\cdot a\rangle} || e^\lambda_{i_o}(k_o k) || e^\lambda_{j_o}(k_o) | \, dk_o \\
&\le\ A \int_K e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dk_o
\end{aligned} \tag{3.7}
$$

where $e_n = (0,\cdots,0,1)^t$ in $\mathbb{R}^n$, $(a,k) \in G$, and A is a constant which depends on λ, i_o, j_o. (Notice that $e^\lambda_{i_o}$ and $e^\lambda_{j_o}$ are continuous functions on K and hence bounded.) Since f satisfies (3.4) and $\beta > \frac{1}{4\alpha}$, we have

$$
| (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | \le\ Ce^{-\beta r^2} \ \le\ Ce^{-\frac{r^2}{4\alpha}},\ r \in \mathbb{R}. \tag{3.8}
$$

By definition of $\Phi^{i_o,j_o}_{r,\lambda}(a,k)$ we have from (3.5),

$$
(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) \ =\ \int_K \int_{\mathbb{R}^n} f(a,k)\Phi^{i_o,j_o}_{r,\lambda}(a,k)\,dadk. \tag{3.9}
$$

Since f satisfies (3.2) and from (3.7), $| \Phi^{i_o,j_o}_{z,\lambda}(a,k) | \le\ Ae^{|z|\|a\|}$, we conclude that the function $r \mapsto (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that $z \mapsto (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ is an entire function. Further, a simple calculation using (3.2) and (3.7) shows that

(3.10)

$$
\begin{aligned}
| (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | \ &\le\ \int_K \int_{\mathbb{R}^n} | f(a,k) || \Phi^{i_o,j_o}_{z,\lambda}(a,k) | \, dadk \\
&\le\ A \int_K \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} \Big(\int_K e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dk_o \Big) dadk \\
&=\ A \int_K \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} e^{-\langle (Im\ z)e_n, k_o\cdot a\rangle} dadk_o \\
&=\ A \int_{\mathbb{R}^n} e^{-\alpha\|a\|^2} e^{-\langle (Im\ z)e_n, a\rangle} da \\
&=\ Ae^{\frac{\|(Im\ z)e_n\|^2}{4\alpha}} \int_{\mathbb{R}^n} e^{-\langle \sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}}, \sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}} \rangle} da \\
&\le\ A' e^{\frac{|Im\ z|^2}{4\alpha}} \\
&\le\ A' e^{\frac{|z|^2}{4\alpha}}
\end{aligned}
$$

for $z \in \mathbb{C}$ and some constants A, A'.

It is clear from (3.8) and (3.10) that the function $z \mapsto (\hat{f}(T_{z,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ satisfies the hypothesis of Lemma 2.1. Hence, it follows that $(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) = Const.e^{-\frac{r^2}{4\alpha}}$. Hence from (3.4) $| (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | = | Const.e^{-\frac{r^2}{4\alpha}} | \le C_\lambda e^{-\beta r^2}$; and since $\beta - \frac{1}{4\alpha} > 0$, we see that $(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) \equiv 0$ as a function of r. Since i_o, j_o and λ were arbitrary, $\hat{f}(T_{r,\lambda}) \equiv 0$ for all $r \in \mathbb{R}^+$ and $\lambda \in SO(n-1)\widehat{\ }$. Hence by the one-to-one property of the group Fourier transform we get that $f = 0$ a.e. This completes the proof of the theorem.

(Actually an examination of the proof shows that we have proved the following stronger result : Let δ_1, $\delta_2 \in \widehat{K}$ and χ_{δ_1} and χ_{δ_2} the corresponding characters. Then $T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})$ is a finite rank operator(with rank bounded by a constant depending only on δ_1, δ_2, λ). This operator is zero on the orthogonal complement of a subspace whose dimension is again bounded by a constant depending only on δ_1, δ_2, λ. Suppose in this context that α and β are positive constants such that $\alpha\beta > \frac{1}{4}$ and that $\mid f(a,k) \mid \leq Ce^{-\alpha\|a\|^2}$ and $\|T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})\|_{HS} \leq C_{\lambda,\delta_1,\delta_2}e^{-\beta r^2}$ where C is a positive constant and $C_{\lambda,\delta_1,\delta_2}$ is a positive constant depending only on δ_1, δ_2, λ. Then $f \equiv 0$.)

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References

1. Cowling M, Price J, *Generalisations of Heisenberg's inequality*, Harmonic analysis, Proceedings, Corotona, Italy(1982), Mauceri G, Ricci F and Weiss G, Eds., Lecture notes No.992, Springer-Verlag, 1983. MR **86g:**42002b
2. Dym H and McKean H P, *Fourier series and integrals*, Academic Press, New York, 1972. MR **56:**945
3. Folland G B, *Abstract harmonic analysis*, Unpublished lecture notes.
4. Gross K I and Kunze R A, *Fourier decompositions of certain representations in "Symmetric Spaces"*, Boothby W and Weiss G, Eds., Marcel-Dekker, New York, 1972, 119-139. MR **55:**572
5. Hörmander L, *A uniqueness theorem of Beurling for Fourier transform pairs*, Arkiv för Matematik, 29(1991), No. 2, 237-240. MR **93b:**42016
6. Pati V, Sitaram A, Sundari M and Thangavelu S, *An uncertainty principle for eigenfunction expansions*, J. Fourier Anal. and Appl., 2(1996). CMP 97:02
7. Sitaram A and Sundari M, *An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups*, Pacific J. Math. 177 (1997), 187–200. CMP 97:11
8. Sitaram A, Sundari M and Thangavelu S, *Uncertainty principles on certain Lie groups*, Proc. Indian Acad. Sci. (Math. Sci.), 105(1995), No. 2, 135-151. MR **96h:**43002
9. Sugiura M, *Unitary representations and harmonic analysis, An introduction*, Kodansha scientific books, Tokyo, 1975. MR **58:**16977
10. Titchmarsh E C, *Introduction to the theory of Fourier Integrals*, Chelsea Publishing Company, New York, N.Y.,1986. MR **89c:**42002

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