

A WEAK-TYPE INEQUALITY OF SUBHARMONIC FUNCTIONS

CHANGSUN CHOI

(Communicated by J. Marshall Ash)

ABSTRACT. We prove the weak-type inequality $\lambda\mu(u + |v| \geq \lambda) \leq (\alpha + 2) \int_{\partial D} u d\mu$, $\lambda > 0$, between a non-negative subharmonic function u and an \mathbb{H} -valued smooth function v , defined on an open set containing the closure of a bounded domain D in a Euclidean space \mathbb{R}^n , satisfying $|v(0)| \leq u(0)$, $|\nabla v| \leq |\nabla u|$ and $|\Delta v| \leq \alpha \Delta u$, where $\alpha \geq 0$ is a constant. Here μ is the harmonic measure on ∂D with respect to 0. This inequality extends Burkholder's inequality in which $\alpha = 1$ and $\mathbb{H} = \mathbb{R}^n$, a Euclidean space.

1. A WEAK-TYPE INEQUALITY

Let Ω be an open subset of \mathbb{R}^n where n is a positive integer. Let D be a bounded subdomain of Ω with $0 \in D$ and $\partial D \subset \Omega$. Let μ be the harmonic measure on ∂D with respect to 0. Let \mathbb{H} be a Hilbert space over \mathbb{R} . For $x, y \in \mathbb{H}$ we denote by $x \cdot y$ the inner product of x and y and put $|x|^2 = x \cdot x$. We consider two smooth functions u and v on Ω ; that is, u and v have continuous partial derivatives up to the second order. Here, u is real-valued and v is \mathbb{H} -valued. By ∇u we denote the gradient of u and by Δu , the Laplacian of u . Write u_i for the partial derivative of u with respect to the i th variable. Thus, $\nabla v = (v_1, \dots, v_n) \in \mathbb{H}^n$, the standard product Hilbert space. Let $\alpha \geq 0$ be a constant.

Theorem. *If u is a non-negative subharmonic function on Ω and*

- (i) $|v(0)| \leq u(0)$,
- (ii) $|\nabla v| \leq |\nabla u|$ on Ω ,
- (iii) $|\Delta v| \leq \alpha \Delta u$ on Ω ,

then, for all $\lambda > 0$, we have

$$\lambda\mu(u + |v| \geq \lambda) \leq (\alpha + 2) \int_{\partial D} u d\mu.$$

Corollary. *If u and v are as in the Theorem, then for all $\lambda > 0$*

$$\lambda\mu(|v| \geq \lambda) \leq (\alpha + 2) \int_{\partial D} u d\mu.$$

Received by the editors May 9, 1996 and, in revised form, October 1, 1996.

1991 *Mathematics Subject Classification.* Primary 31B05.

Key words and phrases. Subharmonic function, smooth function, harmonic measure, weak-type inequality.

This work was partially supported by GARC-KOSEF.

Remark 1.1. In [1] Burkholder proved the inequality in the Theorem when $\alpha = 1$ and $\mathbb{H} = \mathbb{R}^{\nu}$, a Euclidean space. For basic facts of harmonic measures and subharmonic functions one may see [2].

2. TECHNICAL LEMMAS

Put $S = \{(x, y) : x > 0 \text{ and } y \in \mathbb{H} \text{ with } |y| > 0\}$. Define two functions U and V on S by

$$U(x, y) = \begin{cases} (|y| - (\alpha + 1)x)(x + |y|)^{1/(\alpha+1)} & \text{if } x + |y| < 1, \\ 1 - (\alpha + 2)x & \text{if } x + |y| \geq 1 \end{cases}$$

and

$$V(x, y) = \begin{cases} -(\alpha + 2)x & \text{if } x + |y| < 1, \\ 1 - (\alpha + 2)x & \text{if } x + |y| \geq 1. \end{cases}$$

Observe that U is continuous on S .

Lemma 1. (a) $V \leq U$ on S .
 (b) $U(x, y) \leq 0$ if $x \geq |y|$.

Proof. For (a) we may assume $x + |y| < 1$. Write $x + |y| = r^{\alpha+1}$. Since $0 < r < 1$, we have

$$V(x, y) - U(x, y) = -r^{\alpha+2} - (1 - r)(\alpha + 2)x < 0.$$

In order to prove (b) assume $|y| \leq x$. If $x + |y| < 1$, then $|y| - (\alpha + 1)x \leq |y| - x \leq 0$, hence $U(x, y) \leq 0$. If $x + |y| \geq 1$, then $U(x, y) = 1 - (\alpha + 2)x \leq x + |y| - (\alpha + 2)x = |y| - (\alpha + 1)x \leq |y| - x \leq 0$. □

Lemma 2. If $x + |y| < 1$, then $U_x(x, y) + \alpha|U_y(x, y)| \leq 0$.

Proof. If $x + |y| < 1$, then differentiation gives

$$\begin{cases} U_x(x, y) = -\frac{(\alpha + 1)(\alpha + 2)x + \alpha(\alpha + 2)|y|}{(\alpha + 1)(x + |y|)^{\alpha/(\alpha+1)}}, \\ U_y(x, y) = \frac{(\alpha + 2)y}{(\alpha + 1)(x + |y|)^{\alpha/(\alpha+1)}}. \end{cases}$$

Now the lemma is clear. □

For differentiating vector functions one may see [3].

Lemma 3. If $h \in \mathbb{R}$, $k \in \mathbb{H}$, $(x, y) \in S$ and $x + |y| < 1$, then

$$\begin{aligned} &U_{xx}(x, y)h^2 + 2U_{xy}(x, y) \cdot hk + U_{yy}(x, y)k \cdot k \\ &\leq (|k|^2 - h^2) \frac{(\alpha + 2)}{(\alpha + 1)} (x + |y|)^{-\alpha/(\alpha+1)}. \end{aligned}$$

Proof. Put $I = \{t \in \mathbb{R} : x + th > 0, |y + tk| > 0 \text{ and } x + th + |y + tk| < 1\}$ and observe that $0 \in I$ and I is an open set. Define a function G on I by

$$G(t) = U(x + th, y + tk).$$

From the chain rule we have

$$G''(0) = U_{xx}(x, y)h^2 + 2U_{xy}(x, y) \cdot hk + U_{yy}(x, y)k \cdot k.$$

Thus it suffices to show

$$G''(0) \leq (|k|^2 - h^2) \frac{(\alpha + 2)}{(\alpha + 1)} (x + |y|)^{-\alpha/(\alpha+1)}.$$

For this we define more functions K, Q and R on I by $K = K(t) = x + th$, $Q = |y + tk|$ and $R = K + Q$. We omit the argument $t \in I$ in the following computations. On I we have

$$G = R^{(\alpha+2)/(\alpha+1)} - (\alpha + 2)KR^{1/(\alpha+1)}.$$

Differentiating G , we get

$$G' = \frac{\alpha + 2}{\alpha + 1} R' R^{1/(\alpha+1)} - (\alpha + 2)hR^{1/(\alpha+1)} - \frac{\alpha + 2}{\alpha + 1} KR'R^{-\alpha/(\alpha+1)}$$

and

$$\eta G'' = R''R^2 + \frac{1}{\alpha + 1} (R')^2 R - 2hR'R - KR''R + \frac{\alpha}{\alpha + 1} K(R')^2$$

where

$$\eta = \frac{\alpha + 1}{\alpha + 2} R^{(2\alpha+1)/(\alpha+1)}.$$

Rearranging terms and inserting $(R')^2 R - R(R')^2$, we have

$$\begin{aligned} \eta G'' &= (R''R - KR'' - 2hR' + (R')^2)R + \left(-R + \frac{1}{\alpha + 1}R + \frac{\alpha}{\alpha + 1}K\right) (R')^2 \\ &= (|k|^2 - h^2)R - \frac{\alpha}{\alpha + 1}Q(R')^2 \leq (|k|^2 - h^2)R. \end{aligned}$$

Here we used the observation that $K' = h$, $Q' = R' - h$, $QQ' = k \cdot (y + tk)$ and $QR'' = QQ'' = |k|^2 - (Q')^2$. Putting $t = 0$ we get the desired inequality and this proves Lemma 3. \square

Lemma 4. *If $(x_0, y_0) \in S$ and $x_0 + |y_0| = 1$, then $U(x, y) \leq 1 - (\alpha + 2)x$ for all (x, y) in a neighborhood of (x_0, y_0) .*

Proof. Let $(x_0, y_0) \in S$ and $x_0 + |y_0| = 1$. Define $r = r(x, y)$ on S by $x + |y| = r^{\alpha+1}$. Observe that $U(x, y) = 1 - (\alpha + 2)x$ if $r(x, y) \geq 1$. Now assume $0 < r(x, y) < 1$. Then, one can write

$$\begin{aligned} U(x, y) - (1 - (\alpha + 2)x) &= r^{\alpha+2} - (\alpha + 2)xr - (1 - (\alpha + 2)x) \\ &= (r - 1)C(x, y) \end{aligned}$$

where $C(x, y) = (r^{\alpha+2} - 1)/(r - 1) - (\alpha + 2)x$. As (x, y) tends to (x_0, y_0) , we see that $r(x, y)$ tends to 1 and $C(x, y)$ tends to $(\alpha + 2)(1 - x_0) = (\alpha + 2)|y_0| > 0$. Thus the lemma follows.

3. PROOF OF THE INEQUALITY

We may assume $\lambda = 1$ and $\int_{\partial D} u \, d\mu < \infty$; thus we are to prove the inequality

$$(3.1) \quad \mu(u + |v| \geq 1) \leq (\alpha + 2) \int_{\partial D} u \, d\mu.$$

We may further assume that

$$(iv) \quad u > 0 \text{ and } |v| > 0.$$

Indeed, for each $\varepsilon > 0$, the functions $u + \varepsilon$ and (v, ε) , where (v, ε) has value in the standard product Hilbert space $\mathbb{H} \times \mathbb{R}$, satisfy this extra assumption as well as the assumptions of the theorem. Now, the inequality

$$\mu(u + \varepsilon + |(v, \varepsilon)| \geq 1) \leq (\alpha + 2) \int_{\partial D} (u + \varepsilon) d\mu$$

yields, as $\varepsilon \rightarrow 0$, the inequality (3.1) because $\mu(u + |v| \geq 1) \leq \mu(u + \varepsilon + |(v, \varepsilon)| \geq 1)$.

Let the functions U and V be as in the previous section. Observe, from the assumption (iv), that $(u, v) \in S$ on Ω . The inequality (3.1) is equivalent to

$$\int_{\partial D} V(u, v) d\mu \leq 0.$$

According to (a) of Lemma 1 it suffices to prove

$$\int_{\partial D} U(u, v) d\mu \leq 0.$$

Also, (b) of Lemma 1 and the assumption (i) imply $U(u(0), v(0)) \leq 0$. Hence the proof is complete if we can show

$$\int_{\partial D} U(u, v) d\mu \leq U(u(0), v(0))$$

which follows from the superharmonicity of $U(u, v)$.

In order to show that $f = U(u, v)$ is superharmonic on Ω we define subsets Ω^+ and Ω^- of Ω by $\Omega^+ = \{\omega : u(\omega) + |v(\omega)| > 1\}$ and $\Omega^- = \{\omega : u(\omega) + |v(\omega)| < 1\}$. Observe that the continuity of u and v implies that the sets Ω^+ and Ω^- are open.

On Ω^+ we have $f = 1 - (\alpha + 2)u$, which is superharmonic because u is subharmonic by the assumption.

On Ω^- the smooth function f is superharmonic because $\Delta f \leq 0$, as is checked in the following. For each $1 \leq i \leq n$, the chain rule gives

$$f_i = U_x(u, v)u_i + U_y(u, v) \cdot v_i \text{ and } f_{ii} = U_x(u, v)u_{ii} + U_y(u, v) \cdot v_{ii} + A_i$$

where $A_i = U_{xx}(u, v)u_i^2 + 2U_{xy}(u, v) \cdot u_i v_i + U_{yy}(u, v)v_i \cdot v_i$. Thus

$$\Delta f = U_x(u, v)\Delta u + U_y(u, v) \cdot \Delta v + \sum_{i=1}^n A_i.$$

Lemma 2, the assumption (iii), the Cauchy-Schwarz inequality and the assumption that u is subharmonic, imply

$$\begin{aligned} U_x(u, v)\Delta u + U_y(u, v) \cdot \Delta v &\leq U_x(u, v)\Delta u + |U_y(u, v)| |\Delta v| \\ &\leq (U_x(u, v) + \alpha|U_y(u, v)|)\Delta u \leq 0. \end{aligned}$$

On the other hand, for $1 \leq i \leq n$ we put $x = u$, $h = u_i$, $y = v$ and $k = v_i$, and apply the assumption (iv) and Lemma 3 to get

$$\begin{aligned} U_{xx}(u, v)u_i^2 + 2U_{xy}(u, v) \cdot u_i v_i + U_{yy}(u, v)v_i \cdot v_i \\ \leq (|v_i|^2 - u_i^2) \frac{(\alpha + 2)}{(\alpha + 1)} (u + |v|)^{-\alpha/(\alpha+1)}. \end{aligned}$$

Hence

$$\sum_{i=1}^n A_i \leq (|\nabla v|^2 - |\nabla u|^2) \frac{(\alpha + 2)}{(\alpha + 1)} (u + |v|)^{-\alpha/(\alpha+1)} \leq 0$$

by the assumption (ii). This proves that f is superharmonic on Ω^- .

Finally assume $u(\omega) + |v(\omega)| = 1$. Then, by Lemma 4 and the continuity of u and v we have $f \leq 1 - (\alpha + 2)u$ in a neighborhood of ω . Thus, for all small $\rho > 0$ the subharmonicity of u implies

$$\text{Avg}(f; \omega, \rho) \leq \text{Avg}(1 - (\alpha + 2)u; \omega, \rho) \leq 1 - (\alpha + 2)u(\omega) = f(\omega).$$

Here, $\text{Avg}(f; \omega, \rho)$ is the average of f over the ball of radius ρ centered at ω . Hence f is superharmonic at ω .

This proves superharmonicity of $f = U(u, v)$ on Ω and completes the proof of the inequality (3.1), hence that of the Theorem. \square

ACKNOWLEDGEMENT

The author would like to thank Professor D. L. Burkholder for his guidance and kindness during the research of this paper. The author also would like to express his sincere gratitude to the referee who noted typographical errors and whose suggestion resulted in stylistic improvements.

REFERENCES

1. D. L. Burkholder, *Strong differential subordination and stochastic integration*, Ann. Probab. **22** (1994), 995–1025. MR **95h**:60085
2. W. K. Hayman and P. B. Kennedy, *Subharmonic functions*, Academic, New York (1976). MR **57**:665
3. S. Lang, *Analysis I*, Addison-Wesley, Reading, Mass. (1968).

DEPARTMENT OF MATHEMATICS, KAIST, TAEJON 305-701, KOREA
E-mail address: cschoi@math.kaist.ac.kr