PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 983–987
S 0002-9939(98)04170-7

FINITISTIC DIMENSION AND ZIEGLER SPECTRUM

HENNING KRAUSE

(Communicated by Ken Goodearl)

Abstract. Given a two-sided artinian ring Λ, it is shown that the Ziegler spectrum of Λ forms a test class for certain homological properties of Λ. We discuss the finitistic dimension of Λ, Nunke's condition, and also the relation between the big and the little finitistic dimension.

Let Λ be a two-sided artinian ring. Denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and by $\operatorname{mod}\Lambda$ the full subcategory of all finitely presented Λ-modules. Given a Λ-module M, we denote by $\operatorname{pd}M$ its projective dimension, and the *finitistic dimension* $\operatorname{Fin.dim}\Lambda$ of Λ is the supremum of the projective dimensions of the Λ-modules with finite projective dimension. The *Ziegler spectrum* $\operatorname{Zsp}\Lambda$ of Λ is by definition the set of isomorphism classes of indecomposable pure-injective Λ-modules [14].

The aim of this note is to show that the Ziegler spectrum forms a test class for certain homological properties of Λ. Furthermore, the Ziegler spectrum carries a topology, and we shall use its compactness to obtain an equivalent formulation of Nunke's condition. In the final part of this note we study the relation between $\operatorname{Fin.dim}\Lambda$ and the little finitistic dimension $\operatorname{fin.dim}\Lambda$ of Λ.

Our first result is easily stated as follows.

Theorem 1. $\operatorname{Fin.dim}\Lambda = \sup\{\operatorname{pd}M \mid M \in \operatorname{Zsp}\Lambda$ *and* $\operatorname{pd}M < \infty\}$.

We postpone the proof and first discuss some related homological properties of Λ. Suppose that Λ has a self-duality D between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. We are interested in *Nunke's condition* [9]:

$$\operatorname{Nd}M < \infty \text{ for every } \Lambda\text{-module } M \neq 0,$$

where $\operatorname{Nd}M$ denotes the smallest integer n (or ∞ if such an integer does not exist) such that $\operatorname{Ext}^n_\Lambda(D\Lambda, M) \neq 0$. It is convenient to define the *Nunke dimension* of Λ as follows:

$$\operatorname{Nun.dim}\Lambda = \sup\{\operatorname{Nd}M \mid M \in \operatorname{Mod}\Lambda \text{ and } M \neq 0\}.$$

The following relation between Nunke dimension and finitistic dimension is essentially due to Jans [10] (see also [5]).

Proposition 2. $\operatorname{Nun.dim}\Lambda \leq \operatorname{Fin.dim}\Lambda$.

Proof. We use the fact that the functor

$$\operatorname{Inj}\Lambda \longrightarrow \operatorname{Proj}\Lambda, \qquad X \mapsto \operatorname{Hom}_\Lambda(D\Lambda, X)$$

Received by the editors June 7, 1996 and, in revised form, September 25, 1996.
1991 *Mathematics Subject Classification.* Primary 16E10, 16G10.

is an equivalence between the full subcategories of injective and projective Λ-modules. Suppose now that $\operatorname{Nun.dim}\Lambda \geq n$ and choose a non-zero Λ-module M with $\coprod_{i=0}^{n-1}\operatorname{Ext}^i_\Lambda(D\Lambda, M) = 0$. Take an injective coresolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{\psi_0} I_1 \xrightarrow{\psi_1} I_2 \xrightarrow{\psi_2} \dots$$

and put $N = \operatorname{Coker}\operatorname{Hom}_\Lambda(D\Lambda, \psi_{n-1})$. It is easily checked that $\operatorname{pd} N = n$. Thus $\operatorname{Fin.dim}\Lambda \geq n$. □

Remark. Given any $n \in \mathbb{N}_0$, there is a finite dimensional algebra Λ over a field (with $\operatorname{rad}^2\Lambda = 0$ and $\operatorname{Nun.dim}\Lambda = 2$) such that $\operatorname{Fin.dim}\Lambda - \operatorname{Nun.dim}\Lambda = n$ [7].

Theorem 3. $\operatorname{Nun.dim}\Lambda = \sup\{\operatorname{Nd} M \mid M \in \operatorname{Zsp}\Lambda\}$*, and Nunke's condition holds if and only if* $\operatorname{Nun.dim}\Lambda < \infty$.

Some further definitions are needed in order to give the proof of both theorems. An additive functor $F\colon \operatorname{Mod}\Lambda \to \operatorname{Ab}$ into the category of abelian groups is *finitely presented* if there is a presentation

$$\operatorname{Hom}_\Lambda(Y, \ ) \longrightarrow \operatorname{Hom}_\Lambda(X, \ ) \longrightarrow F \longrightarrow 0$$

with X and Y in $\operatorname{mod}\Lambda$. We denote by $\operatorname{Ker} F$ the full subcategory of Λ-modules M with $F(M) = 0$. A full subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ is *definable* if there is a family F_i, $i \in I$, of finitely presented functors such that $\mathcal{X} = \bigcap_{i\in I}\operatorname{Ker} F_i$. A definable subcategory is automatically closed under direct limits, products and pure submodules, and it can be shown that this property characterizes a definable subcategory [12]. The notion of a definable subcategory allows us to describe the topology on $\operatorname{Zsp}\Lambda$ which was first introduced by Ziegler in model theoretic terms [14].

Proposition 4. (1) *The assignment* $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ *defines a bijection between the definable subcategories of* $\operatorname{Mod}\Lambda$ *and the closed subsets of* $\operatorname{Zsp}\Lambda$.

(2) $\operatorname{Zsp}\Lambda$ *is a quasi-compact space.*

Proof. (1) This result is due to Crawley-Boevey [3], and he relies on work of Herzog [6] and Ziegler [14]. We refer to [12] for a proof which uses the localization theory for locally coherent categories developed in [11]. However, the inverse of the assignment $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ is easily constructed; it sends a closed subset $\mathcal{U}$ of $\operatorname{Zsp}\Lambda$ to $\bigcap_{F(\mathcal{U})=0}\operatorname{Ker} F$.

(2) See [14] or [11]. □

The next lemma shows that some relevant subcategories of $\operatorname{Mod}\Lambda$ are definable.

Lemma 5. *Let* $n \in \mathbb{N}_0$.

(1) *The full subcategory* $\mathcal{P}_n$ *of* Λ*-modules* M *with* $\operatorname{pd} M \leq n$ *is definable.*

(2) *The full subcategory* $\mathcal{Q}_n$ *of* Λ*-modules* M *with* $\operatorname{Nd} M \geq n$ *is definable.*

Proof. Projective dimension and flat dimension coincide for any module over an artinian ring, and therefore $\mathcal{P}_n = \bigcap_{X\in\operatorname{mod}\Lambda^{\mathrm{op}}}\operatorname{Ker}\operatorname{Tor}^\Lambda_{n+1}(\ , X)$ since $\operatorname{Tor}^\Lambda_{n+1}(M, \ )$ commutes with direct limits and every module can be written as a direct limit of finitely presented modules. Therefore $\mathcal{P}_n$ is definable, and $\mathcal{Q}_n$ is definable since $\mathcal{Q}_n = \operatorname{Ker}\coprod_{i=0}^{n-1}\operatorname{Ext}^i_\Lambda(D\Lambda, \ )$. □

An easy consequence of the preceding lemma is the following.

Proposition 6. *Let* $\mathcal{U}$ *be a subset of* $\operatorname{Zsp}\Lambda$. *Then* $\operatorname{pd} M \leq \sup\{\operatorname{pd} N \mid N \in \mathcal{U}\}$ *for every module* M *in the closure of* $\mathcal{U}$.

Proof of Theorem 1. Suppose that $\mathcal{P}_n$ is properly contained in $\mathcal{P}_{n+1}$. The first part of Proposition 4 then implies there is $M \in \operatorname{Zsp}\Lambda$ with $M \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$, since $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ are definable. Thus the assertion follows. □

Proof of Theorem 3. The proof of the first part of the assertion is analogous to that of Theorem 1. Suppose now that Nunke's condition holds. Thus $\bigcap_{n\geq 0} \mathcal{Q}_n = 0$, and the compactness of $\operatorname{Zsp}\Lambda$ then implies the existence of $d \in \mathbb{N}_0$ with $\bigcap_{n=0}^{d} \mathcal{Q}_n = 0$. We conclude that $\operatorname{Nun.dim}\Lambda < d$. □

Remark. The finitistic dimension of Λ is finite if and only if the full subcategory of all Λ-modules with finite projective dimension is definable. Therefore one might conjecture that $\operatorname{Fin.dim}\Lambda < \infty$ if and only if the modules in the Ziegler spectrum having finite projective dimension form a closed subset of $\operatorname{Zsp}\Lambda$.

The rest of this paper is devoted to studying the relation between the finitistic dimension $\operatorname{Fin.dim}\Lambda$ and the *little finitistic dimension* $\operatorname{fin.dim}\Lambda$, which is the supremum of the projective dimensions of the finitely presented Λ-modules with finite projective dimension. Note that there are examples of finite dimensional algebras where $\operatorname{fin.dim}\Lambda$ is different from $\operatorname{Fin.dim}\Lambda$ [15]. Our results are motivated by recent work of Huisgen-Zimmermann and Smalø, who have shown that both dimensions coincide for an interesting class of Artin algebras [8].

In order to state the next result we recall from [2] that for any subcategory $\mathcal{C}$ of $\operatorname{Mod}\Lambda$ a morphism $M \to M'$ is a *left* $\mathcal{C}$*-approximation* of M if M' lies in $\mathcal{C}$ and the induced map $\operatorname{Hom}_\Lambda(M', X) \to \operatorname{Hom}_\Lambda(M, X)$ is surjective for every X in $\mathcal{C}$. We denote by $\mathcal{P}_\infty$ the full subcategory of all Λ-modules with finite projective dimension.

Theorem 7. *The following conditions are equivalent.*

(1) *A* Λ*-module has finite projective dimension if and only if it is a direct limit of finitely presented modules having finite projective dimension.*

(2) *The little finitistic dimension* $\operatorname{fin.dim}\Lambda$ *is finite, and every finitely presented* Λ*-module has a left* $\mathcal{P}_\infty$*-approximation which is finitely presented.*

Moreover, if (1)–(2) *hold, then* $\operatorname{fin.dim}\Lambda = \operatorname{Fin.dim}\Lambda < \infty$.

Examples of rings which satisfy the conditions in this theorem are Artin algebras Λ where every finitely presented Λ-modules has a right $\mathcal{P}_\infty \cap \operatorname{mod}\Lambda$-approximation [1, 8]. There are also examples of Artin algebras Λ which show that in condition (2) the term "$\mathcal{P}_\infty$-approximation" cannot be replaced by "$\mathcal{P}_\infty \cap \operatorname{mod}\Lambda$-approximation" [13]. The proof of Theorem 7 is based on the following lemma.

Lemma 8. *Let* $\mathcal{C}$ *be any additive subcategory of* $\operatorname{mod}\Lambda$. *Then a* Λ*-module* M *is a direct limit of objects in* $\mathcal{C}$ *iff every morphism* $X \to M$ *with* $X \in \operatorname{mod}\Lambda$ *factors through an object in* $\mathcal{C}$.

Proof. See [4, Lemma 4.1]. □

Proof of Theorem 7. (1) ⇒ (2) In order to show that $\operatorname{fin.dim}\Lambda < \infty$ assume there is a family M_i, $i \in \mathbb{N}$, in $\operatorname{mod}\Lambda$ with $\operatorname{pd} M_i < \operatorname{pd} M_{i+1} < \infty$ for all i. Then $\coprod_{i\in\mathbb{N}} M_i$ is a direct limit of finitely presented objects in $\mathcal{P}_\infty$, but $\operatorname{pd} \coprod_{i\in\mathbb{N}} M_i = \infty$. Thus (1) implies $\operatorname{fin.dim}\Lambda = n < \infty$ and therefore $\operatorname{Fin.dim}\Lambda = n$, since $\mathcal{P}_n$ is closed under direct limits by Lemma 5. To show the second part of the assertion we use an idea from [4]. Let M be in $\operatorname{mod}\Lambda$ and choose a representative set $\varphi_i\colon M \to N_i$, $i \in I$, of morphisms in $\operatorname{mod}\Lambda$ such that the domain has finite projective dimension. Then $N = \prod_{i\in I} N_i$ lies in $\mathcal{P}_\infty$, since $\mathcal{P}_\infty = \mathcal{P}_n$ is closed under products by Lemma 5. Thus the induced morphism $\varphi\colon M \to N$ factors through a finitely presented object M' in $\mathcal{P}_\infty$ by the preceding lemma. It is easily checked that the corresponding morphism $M \to M'$ is a left $\mathcal{P}_\infty$-approximation of M.

(2) ⇒ (1) It follows immediately from the preceding lemma that every module in $\mathcal{P}_\infty$ is a direct limit of finitely presented modules in $\mathcal{P}_\infty$. Conversely, the projective dimension of a direct limit of finitely presented Λ-modules in $\mathcal{P}_\infty$ is bounded by $\operatorname{fin.dim}\Lambda = n$, since $\mathcal{P}_n$ is closed under direct limits by Lemma 5. □

If the ring Λ has the property that every finitely presented Λ-module is pure-injective, for instance if Λ is an Artin algebra, then condition (1) in the preceding theorem implies that the finitely presented modules form a dense subset of $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\}$. We point out that this condition is already sufficient for the equality $\operatorname{fin.dim}\Lambda = \operatorname{Fin.dim}\Lambda$.

Corollary 9. *Let $n \in \mathbb{N}_0$ and suppose that the finitely presented modules $M \in \operatorname{Zsp}\Lambda$ with $\operatorname{pd} M \leq n$ form a dense subset of $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\}$. Then $\operatorname{fin.dim}\Lambda = \operatorname{Fin.dim}\Lambda \leq n$.*

Proof. Let $\mathcal{U}_n = \{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M \leq n \text{ and } M \in \operatorname{mod}\Lambda\}$, and we may assume that $n \in \mathbb{N}_0$ is the minimal choice with $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\} \subseteq \overline{\mathcal{U}_n}$. It follows from Proposition 6 and Theorem 1 that $\operatorname{Fin.dim}\Lambda \leq n$, and we therefore obtain $n \leq \operatorname{fin.dim}\Lambda \leq \operatorname{Fin.dim}\Lambda \leq n$. Thus $\operatorname{fin.dim}\Lambda = \operatorname{Fin.dim}\Lambda = n$. □

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Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`