

COMMON ZEROS OF THETA FUNCTIONS AND CENTRAL HECKE L-VALUES OF CM NUMBER FIELDS OF DEGREE 4

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ABSTRACT. In this note, we apply the method of Rodriguez Villegas and Yang (1996) to construct a family of infinite many theta series over the Hilbert-Blumenthal modular surfaces with a common zero. We also relate the nonvanishing of the central L-values of certain Hecke characters of non-biquadratic CM number fields of degree 4 to the nonvanishing of theta functions at CM points in the Hilbert-Blumenthal modular surfaces.

0. INTRODUCTION

It is well-known that the theta function $\theta(z) = \sum e(x^2 z)$ has no zeros in the upper half plane, where $e(z) = e^{2\pi iz}$. It was proved in [RVY] that, for integers $d \equiv 1 \pmod{4}$ and $k \geq 0$, whether the theta functions

$$\theta_{d,k}(z) = (\operatorname{Im}(2z))^{-\frac{k}{2}} \sum_{(x,d)=1} \left(\frac{d}{x}\right) H_k(x\sqrt{\operatorname{Im}(2z)}) e(x^2 z)$$

vanish at certain Heegner points is related to the vanishing of central Hecke L-values ([RVY, Theorem 0.2]). Here H_k is a suitably normalized Hermite polynomial. Over a real quadratic field $F = \mathbb{Q}(\sqrt{p})$, one can also define a theta function $\theta_{d,k}$ on the Hilbert upper half plane \mathcal{H}^2 in the same manner. In fact, one can replace $\left(\frac{d}{x}\right)$ by any quadratic Hecke character of F . (See (1) and (10) for the exact meaning). They are nonholomorphic Hilbert modular forms of weight $k + \frac{1}{2}$ for some congruence group Γ_d over F . We remark that Γ_d is independent of k . The first result involves the zero locus of these theta functions. We prove that those theta functions sometimes have a common zero when k varies.

Theorem 0. *Let $p = 5, 13, \text{ or } 61$. Let $d \equiv 1 \pmod{4}$ be a rational integer such that every prime factor of d is split in either $\mathbb{Q}(\sqrt{-7})$ or $\mathbb{Q}(\sqrt{-7p})$.*

(a) *If $d < 0$, then $\{\theta_{d,k} : k \geq 0, \text{ even}\}$ has a common zero in $\mathbb{Q}(\sqrt{p}, \sqrt{-7}) \cap \mathcal{H}^2$.*

(b) *If $d > 0$, then $\{\theta_{d,k} : k \geq 1, \text{ odd}\}$ has a common zero in $\mathbb{Q}(\sqrt{p}, \sqrt{-7}) \cap \mathcal{H}^2$.*

Here we embed $\mathbb{Q}(\sqrt{p}, \sqrt{-7})$ into \mathbb{C}^2 via $z \mapsto (\sigma_1(z), \sigma_2(z))$, where $\sigma_j(\sqrt{p}) = (-1)^{j-1} \sqrt{p}$ and $\sigma_j(\sqrt{-7}) = i\sqrt{7}$ for $j = 1, 2$.

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The theorem is also true if we replace p by 17 and -7 by -11 . The proof of this theorem and its generalization will be given in section 2. In this note, we will also relate the central Hecke L-value to the values of theta functions at CM points and relate the vanishing of the Hecke L-value to the CM zeros of the theta functions (Theorem 2 and 8).

1. NOTATION AND PRELIMINARIES

Assume $F = \mathbb{Q}(\sqrt{p})$ is a real quadratic number field with $p \equiv 1 \pmod{4}$. We view F as a subfield of \mathbb{R} with $\sqrt{p} > 0$, so the other embedding of F into \mathbb{R} is given by $x = a + b\sqrt{p} \mapsto x' = a - b\sqrt{p}$, $a, b \in \mathbb{Q}$. We embed F into \mathbb{C}^2 via $x \mapsto (x, x')$. Let $\mathcal{H}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_i > 0\}$ be the Hilbert upper plane. Given a function ϕ on \mathbb{C} , we define a function on \mathbb{C}^2 , still denoted by ϕ , via $\phi(z) = \phi(z_1)\phi(z_2)$. Let u be a fixed unit of F with $u > 0$ and $u' < 0$. Let μ be a quadratic Hecke character of F with principal conductor $f\mathcal{O}_F$; then μ induces a character $\mu : (\mathcal{O}_F/f)^* \rightarrow \mathbb{C}^1$. We assume $\mu(-1) = 1$. Multiplying by u if necessary, we may assume that f is totally positive. For a rational integer $k \geq 0$, we define a theta function $\theta_{\mu,k}$ on the Hilbert upper plane \mathcal{H}^2 via

$$(1) \quad \theta_{\mu,k}(z) = (\text{Im}(2z))^{-\frac{k}{2}} \sum_{\substack{x \in \mathcal{O}_F \\ (x,f)=1}} \mu(x)H_k(x\sqrt{\text{Im}(2z)})e(x^2z).$$

Here $e(z) = e^{2\pi iz}$, and H_k is the k th Hermite polynomial normalized via [RVY, (0.13)]. The theta function $\theta_{\mu,k}$ is a nonholomorphic Hilbert modular form of weight $k + \frac{1}{2}$ for some congruence group Γ_d over F and is holomorphic when $k = 0$ or 1. We notice that Γ_d is independent of k .

Let D be a totally positive integer in \mathcal{O}_F , and let $E = F(\sqrt{-D})$ be a totally imaginary extension of F . We assume throughout this paper

- (2) $D\mathcal{O}_F$ is prime, and E/F is ramified only at $D\mathcal{O}_F$,
- (3) every prime of F dividing $2f\mathcal{O}_F$ is split in E/F , and
- (4) E is generic in the sense of Rohrlich ([Roh, page 519]), that is, $\mathcal{O}_E^* = \mathcal{O}_F^*$ and $\text{CL}(F) \hookrightarrow \text{CL}(E)$ naturally.

Let $\Phi = \{\sigma_1, \sigma_2\}$ be the CM type of E given by

$$\sigma_1(\sqrt{p}) = \sqrt{p}, \quad \sigma_1(\sqrt{-D}) = i\sqrt{D}$$

and

$$\sigma_2(\sqrt{p}) = -\sqrt{p}, \quad \sigma_2(\sqrt{-D}) = i\sqrt{D}'.$$

Here \sqrt{a} stands for the positive square root of a for $a > 0$. We embed E into \mathbb{C}^2 via $z \mapsto (\sigma_1(z), \sigma_2(z))$.

Let $\tilde{\mu} = \mu \circ N_{E/F}$; then $\tilde{\mu}|_{F_{\mathbb{A}}^*} = 1$ and there is a character η of $E^1 \backslash E_{\mathbb{A}}^1$ such that $\tilde{\mu} = \tilde{\eta}$, where $\tilde{\eta}(z) = \eta(z/\bar{z})$. Let χ_{can} be a canonical Hecke character of E of infinite type Φ in the sense of Rohrlich ([Roh, page 518]). Then the character $\chi_{\text{can}}\tilde{\mu}$ is closely related to CM abelian varieties ([Sh]) and has some simple properties ([RVY, Lemma 2.0]). Let $\chi = \chi_{\text{can}}|_{F_{\mathbb{A}}^{\frac{1}{2}}}$, then $\chi|_{F_{\mathbb{A}}^*} = \epsilon$ is the quadratic Hecke character of F associated to E/F . Denote $\eta_k = \eta\chi^k|_{E_{\mathbb{A}}^1}$, so $\tilde{\eta}_k = \tilde{\eta}\chi^{2k}$. Let $\psi = \prod \psi_v$ be the ‘canonical’ additive character of $F_{\mathbb{A}}/F$ given by

$$\psi_v(x) = \begin{cases} e^{2\pi ix} & \text{if } v \text{ is real,} \\ e^{-2\pi i\lambda(x)} & \text{if } v \text{ is finite,} \end{cases}$$

where $\lambda : F_v \xrightarrow{\text{tr}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Let $\psi_E = \psi \circ \text{tr}_{E/F}$.

Lemma 1. *Let $\alpha = \frac{4u}{D\sqrt{p}}$ and $\delta = \sqrt{-D}$. Then for any rational integer $k \geq 0$, the datum $(\chi, \eta_k, \alpha, \delta, \psi, \Phi)$ satisfies (0.4), (0.6), and (0.9) in [RVY].*

Proof. It is easy to see that

$$\epsilon_v(\alpha) = \begin{cases} \epsilon_{\sqrt{p}}(\sqrt{p}) & \text{if } v = \sqrt{p}\mathcal{O}_F \text{ or } D\mathcal{O}_F, \\ 1 & \text{otherwise.} \end{cases}$$

Here we have abused the notation by using \sqrt{p} for the place of F corresponding to the prime ideal $\sqrt{p}\mathcal{O}_F$. Since $n(\psi_v) = -1$ when $v = \sqrt{p}\mathcal{O}_F$ and $n(\psi_v) = 0$ otherwise, one has, by [RVY, Lemma 2.3],

$$\epsilon\left(\frac{1}{2}, (\chi\tilde{\eta}_k)_v, \frac{1}{2}\psi_{E_v}\right)(\chi\tilde{\eta}_k)_v(\delta) = \begin{cases} ? & \text{if } v = D\mathcal{O}_F, \\ \epsilon_{\sqrt{p}}(\sqrt{p}) & \text{if } v = \sqrt{p}\mathcal{O}_F, \\ 1 & \text{otherwise.} \end{cases}$$

But the global root number $\epsilon(\frac{1}{2}, \chi\tilde{\eta}_k) = \mu(-1)(-1)^{2k} = 1$ by [RVY, Lemma 2.3]. So $? = \epsilon_{\sqrt{p}}(\sqrt{p})$. Therefore

$$(5) \quad \epsilon\left(\frac{1}{2}, (\chi\tilde{\eta}_k)_v, \frac{1}{2}\psi_{E_v}\right)(\chi\tilde{\eta}_k)_v(\delta) = \epsilon_v(\alpha)$$

for every place v of F . This verifies condition (0.6) in [RVY]. The verification of [RVY, (0.4) and (0.9)] is trivial. \square

Now applying [RVY, Theorem 2.7], one has

Theorem 2. *Notation as above. Assume that F has ideal class number 1. Fix a square root r of $-D \pmod{16f^2}$. For every ideal class $C \in \text{CL}(E)$, choose a primitive ideal $\mathfrak{A} \in C^{-1}$ relatively prime to $2f$, and write*

$$(6) \quad \mathfrak{A}^2 = \left[a^2, \frac{b + \sqrt{-D}}{2} \right]$$

with a totally positive, and

$$(7) \quad b \equiv r \pmod{8f^2}.$$

Then

$$(8) \quad L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = \kappa \left| \sum_{C \in \text{CL}(E)} \frac{\mu(a)}{(\chi_{\text{can}}\tilde{\mu})^{2k+1}(\mathfrak{A})} \theta_{\mu, k}(\tau_{\mathfrak{A}}) \right|^2.$$

Here

$$\kappa = \frac{\pi^{2k+2}}{(k!)^2 \sqrt{N_{F/\mathbb{Q}} D}} \left(N_{F/\mathbb{Q}} \frac{D}{pf^4} \right)^{\frac{2k+1}{4}},$$

and

$$\tau_{\mathfrak{A}} = \frac{u(b + \sqrt{-D})}{8\sqrt{p}f^2a^2}.$$

For a CM type Φ of E , we write $k\Phi$ for $\sum_{\sigma \in \Phi} k\sigma$. According to Rohrlich ([Roh2, page 700]), there is a group homomorphism

$$(9) \quad h_{k\Phi} : \text{CL}(E) \longrightarrow \text{CL}(E^{k\Phi}).$$

Here $E^{k\Phi}$ is the number field generated by $z^{k\Phi} = \prod \sigma(z)^k, z \in E^*$. Applying [RVY, Theorem 2.8], one has

Corollary 3. *Notation and assumption as in Theorem 2. Assume further that $h_{(2k+1)\Phi}$ is injective. Then the following statements are equivalent.*

- (a) *The central L-value $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$.*
- (b) *The global theta lifting $\theta_{\alpha, \chi}(\eta_k) = 0$.*
- (c) *For every ideal class $C \in \text{CL}(E)$, $I_C(\eta_k) = 0$.*
- (d) *For every ideal class $C \in \text{CL}(E)$, $\tau_{\mathfrak{A}}$ is a root of the theta function $\theta_{\mu, k}$.*

In general, one has

Proposition 4. *Notation as in Lemma 1. If $\text{CL}(F) \supset \ker h_{(2k+1)\Phi}$, and the central L-value $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$, then $\tau = \frac{u(b+\sqrt{-D})}{8\sqrt{pf^2}} \in E$ is a root of the Hilbert modular form $\theta_{\mu, k}$. Here $b \in \mathcal{O}_F$ satisfies $b^2 \equiv -D \pmod{16f^2}$.*

Proof. Applying the argument in the proof of [RVY, Theorem 2.8] to [RVY, Theorem 2.6], one has that $2u(b + \sqrt{-D})$ is a root of $\theta_{\mu, k, \mathcal{O}_F}$, where $\theta_{\mu, k, \mathcal{O}_F}$ is given by [RVY, (2.3)]. Simple manipulation gives

$$\theta_{\mu, k, \mathcal{O}_F}(z) = \gamma' N_{F/\mathbb{Q}}(4f)^{-k} \theta_{\mu, k}\left(\frac{z}{16f^2}\right)$$

where $\gamma' = \prod_{v|f} \frac{G(\frac{1}{4f}\psi_v, \mu_v)}{\sqrt{q_v}} \in \mathbb{C}^1$. So τ is a root of $\theta_{\mu, k}$. □

2. COMMON ZEROS OF THETA FUNCTIONS

Recall that $F = \mathbb{Q}(\sqrt{p})$, with $p \equiv 1 \pmod{4}$ being a prime. Let $d \equiv 1 \pmod{4}$ be a rational integer relatively prime to p , and let ϵ^d be the Hecke character of \mathbb{Q} associated to the Dirichlet character $(\frac{d}{\cdot})$, and let $\mu = \epsilon^d \circ N_{F/\mathbb{Q}}$. Denote $\theta_{d, k} = \theta_{\mu, k}$, i.e.,

$$(10) \quad \theta_{d, k}(z) = (\text{Im}(2z))^{-\frac{k}{2}} \sum_{x \in \mathcal{O}_F} \left(\frac{d}{xx'}\right) H_k(x\sqrt{\text{Im}(2z)}) e(x^2 z).$$

For a rational prime number $q \equiv 3 \pmod{4}$, let $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ and $F_2 = \mathbb{Q}(\sqrt{-q})$. For an integer $k \geq 0$, let $N^k : \text{CL}(E) \longrightarrow \text{CL}(F_2)$ be the map induced by $\mathfrak{A} \mapsto (N_{E/F_2}(\mathfrak{A}))^{2k+1}$.

Proposition 5. *Let $q \equiv 3 \pmod{4}$ be a rational prime number such that $(q/p) = -1$ and every prime factor of $2d$ is split in either $\mathbb{Q}(\sqrt{-q})$ or $\mathbb{Q}(\sqrt{-pq})$. Assume further that $q > 3$ and $\text{CL}(F) \supset \ker N^k$. Let $b \in \mathcal{O}_F$ be an algebraic integer in F such that $b^2 \equiv -q \pmod{16d^2}$.*

- (a) *If $d < 0$ and k is even, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a root of $\theta_{d, k}$.*
- (b) *If $d > 0$ and k is odd, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a root of $\theta_{d, k}$.*

Proof. Since p is a prime, $\text{CL}(F)$ is an odd group, and so the canonical map $\text{CL}(F) \longrightarrow \text{CL}(E)$ is injective. Since $q > 3$, $\mathcal{O}_E^* = \mathcal{O}_F^*$. So E is generic. Let $\chi_{\text{can}, 2}$ be a canonical Hecke character of F_2 , and let $\chi_{\text{can}} = \chi_{\text{can}, 2} \circ N_{E/F_2}$; then

χ_{can} is a canonical Hecke character of E of infinite type $\Phi = \{\sigma_1, \sigma_2\}$. Moreover, one has

$$L(s, \chi_{\text{can}}^{2k+1}(\frac{d}{-}) \circ N_{E/\mathbb{Q}}) = L(s, \chi_{\text{can},2}^{2k+1}(\frac{d}{-}) \circ N_{F_2/\mathbb{Q}})L(s, (\chi_{\text{can},2}^{2k+1}(\frac{d}{-}) \circ N_{F_2/\mathbb{Q}})^{\sigma_2}).$$

Under either condition (a) or (b), the global root number of $\chi_{\text{can},2}^{2k+1}(\frac{d}{-}) \circ N_{F_2/\mathbb{Q}}$ is -1 , and so the central L-value $L(k+1, \chi_{\text{can},2}^{2k+1}(\frac{d}{-}) \circ N_{F_2/\mathbb{Q}}) = 0$. This implies $L(k+1, \chi_{\text{can}}^{2k+1}(\frac{d}{-}) \circ N_{E/\mathbb{Q}}) = 0$. Notice that $h_{(2k+1)\Phi} = N^k$ in this case. Now applying Proposition 4, one proves the proposition. \square

Corollary 6. *Notation as in Proposition 5. Assume $q > 3$ and $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ has ideal class number 1.*

(a) *If $d < 0$, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a common root of $\theta_{d,k}$ for all even integers $k \geq 0$.*

(b) *If $d > 0$, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a common root of $\theta_{d,k}$ for all odd integers $k > 0$.*

Proof of Theorem 0. According to [Yam, Table], the biquadratic CM fields satisfying all the conditions in Corollary 6 are $\mathbb{Q}(\sqrt{p}, \sqrt{-7})$ with $p = 5, 13, 61$ and $\mathbb{Q}(\sqrt{17}, \sqrt{-11})$. This proves Theorem 0.

3. NONBIQUADRATIC CM NUMBER FIELDS OF DEGREE 4

Let the notation and assumptions be as in section 1. We further assume that D is not a rational integer. Write $D = x + y\sqrt{p} > 0$ with $x, y \in \frac{1}{2}\mathbb{Z}$, $x \equiv y \pmod{\mathbb{Z}}$, and $y \neq 0$. Then $D' = x - y\sqrt{p} > 0$ and $q = DD'$ is a positive rational prime number. When $q = p$, E is cyclic over \mathbb{Q} and is a subfield of $\mathbb{Q}(\zeta_p)$. In this case, $E^\Phi = E$. When $q \neq p$, let $K = E(\sqrt{-D'})$, and identify it as a subfield of \mathbb{C} via

$$\sigma_1 : K \longrightarrow \mathbb{C}; \quad \sqrt{p} \mapsto \sqrt{p}, \quad \sqrt{-D} \mapsto i\sqrt{D}, \quad \sqrt{-D'} \mapsto i\sqrt{D'}.$$

Let

$$\delta = \sqrt{-D} = i\sqrt{D}, \quad \delta' = \sqrt{-D'} = i\sqrt{D'},$$

and

$$\tilde{\delta} = \delta + \delta', \quad \tilde{\delta}' = \delta - \delta'.$$

Then $\delta\delta' = -q$ and $\tilde{\delta}\tilde{\delta}' = -2yp$. Furthermore, let $E' = F(\delta')$, $\tilde{F} = \mathbb{Q}(\sqrt{q})$, $\tilde{E} = \tilde{F}(\tilde{\delta})$, and $\tilde{E}' = \tilde{F}(\tilde{\delta}')$. Then E' , \tilde{E} , and \tilde{E}' are all subfields of K , and K is a dihedral extension of \mathbb{Q} of degree 8. Write $\text{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau \rangle$ with

$$\sigma : \delta \mapsto \delta', \quad \delta' \mapsto -\delta, \quad \tilde{\delta} \mapsto -\tilde{\delta}', \quad \tilde{\delta}' \mapsto \tilde{\delta}, \quad \sqrt{p} \mapsto -\sqrt{p}, \quad \sqrt{q} \mapsto -\sqrt{q},$$

and

$$\tau : \delta \mapsto \delta, \quad \delta' \mapsto -\delta', \quad \tilde{\delta} \mapsto \tilde{\delta}', \quad \tilde{\delta}' \mapsto \tilde{\delta}, \quad \sqrt{p} \mapsto \sqrt{p}, \quad \sqrt{q} \mapsto -\sqrt{q}.$$

Notice that $\Phi = \{1, \sigma\}$. Let $\tilde{\Phi} = \{1, \tau\}$; then $\tilde{\Phi}$ is a CM type of \tilde{E} . One has $E^\Phi = \tilde{E}$ and $\tilde{E}^{\tilde{\Phi}} = E$. So for every integer $n > 0$ there is a map

$$h_{n\tilde{\Phi}} \circ h_{n\Phi} : \text{CL}(E) \longrightarrow \text{CL}(\tilde{E}) \longrightarrow \text{CL}(E).$$

Proposition 7. *Assume that F has ideal class number 1.*

- (a) *When $q \neq p$, the map $h_{n\tilde{\Phi}} \circ h_{n\Phi}$ is the $(2n)$ -th power map.*
- (b) *When $q = p$, $h_{n\tilde{\Phi}}^2$ is the $(2n)$ -th power map.*

Proof. Given an ideal \mathfrak{A} of E , we also use \mathfrak{A} for its ideal class in $\text{CL}(E)$. We only prove the case $n = 1$, as the others are the same. By definition ([Roh2, page 700]), $h_{\Phi}(\mathfrak{A}) = (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap \mathcal{O}_{\bar{E}}$. Since $\text{Gal}(K/\bar{E}) = \{1, \tau\sigma^3\} = \{1, \sigma\tau\}$, and $\tau \in \text{Gal}(K/E)$, one has

$$\begin{aligned} h_{\Phi}(\mathfrak{A})\mathcal{O}_K &= (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap (\tau\sigma^3(\mathfrak{A})\tau(\mathfrak{A})\mathcal{O}_K) \\ &= (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap (\sigma\tau(\mathfrak{A})\mathfrak{A}\mathcal{O}_K) \\ &= \mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K. \end{aligned}$$

So

$$\begin{aligned} h_{\bar{\Phi}} \circ h_{\Phi}(\mathfrak{A}) &= (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K)^{\bar{\Phi}} \cap E \\ &= \mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K\tau(\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap E \\ &= \mathfrak{A}^2\sigma(\mathfrak{A})\tau\sigma(\mathfrak{A})\mathcal{O}_K \cap E \\ &= \mathfrak{A}^2\sigma(\mathfrak{A})\sigma^3(\mathfrak{A})\mathcal{O}_K \cap E \\ &= \mathfrak{A}^2N_{E'/F}(\sigma(\mathfrak{A})) \\ &= \mathfrak{A}^2 \end{aligned}$$

since F has ideal class number 1. This proves (a). Claim (b) can be proved similarly. \square

Now applying Corollary 3, we immediately get

Theorem 8. *Assume that F has ideal class number 1, and that $E = F(\sqrt{-D})$ satisfies conditions (2)–(4). Assume further that D is not a rational number and $(2(2k+1), h_E) = 1$, where h_E is the ideal class number of E . Then the central L -value*

$$L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$$

if and only if for every ideal class $C \in \text{CL}(E)$ and a (and every) primitive ideal $\mathfrak{A} \in C^{-1}$ relatively prime to $2f$, the CM point $\tau_{\mathfrak{A}}$ in Theorem 2 is a root of the Hilbert modular form $\theta_{\mu, k}$.

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