

A TWISTOR CORRESPONDENCE AND PENROSE TRANSFORM FOR ODD-DIMENSIONAL HYPERBOLIC SPACE

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ABSTRACT. For odd-dimensional hyperbolic space \mathcal{H} , we construct transforms between the cohomology of certain line bundles on \mathcal{T} (a twistor space for \mathcal{H}) and eigenspaces of the Laplacian Δ and of the Dirac operator D on \mathcal{H} . The transforms are isomorphisms. As a corollary we obtain that every eigenfunction of Δ or D on \mathcal{H} extends as a holomorphic eigenfunction of the corresponding holomorphic operator on a certain region of the complexification of \mathcal{H} . We also obtain vanishing theorems for the cohomology of a class of line bundles on \mathcal{T} .

1. INTRODUCTION

The group of direct isometries of real hyperbolic $(2n + 1)$ -space \mathcal{H} is the identity component $G = SO_0(2n + 1, 1)$ of the pseudo-orthogonal group. Its universal covering group is $\tilde{G} = \text{Spin}_0(2n + 1, 1)$. We consider another homogeneous space G/L which has a natural complex structure as a consequence of its being identified with an open orbit in a generalised flag variety $G^{\mathbb{C}}/Q$, and show that there is a G -equivariant correspondence whereby \mathcal{H} parameterises a family of $\frac{n(n+1)}{2}$ -dimensional compact complex submanifolds of G/L . We then apply the Penrose transform techniques described in [BES] to prove isomorphisms between certain holomorphic cohomology groups of \tilde{G} -homogeneous holomorphic line bundles on G/L and kernels of differential operators on \mathcal{H} . The \tilde{G} -homogeneous holomorphic line bundles on G/L are described by an integer-valued parameter k and a complex-valued parameter λ . For $k = -2n$, we obtain an isomorphism between the $\frac{n(n+1)}{2}$ -th cohomology and the set of all smooth functions ϕ on \mathcal{H} satisfying $\Delta\phi = (\lambda^2 - n^2)\phi$, where Δ denotes the Laplacian, and in a similar way when $k = -2n - 1$ we obtain all solutions of $D\sigma = -i\lambda\sigma$, where σ is a smooth section of the spin bundle over \mathcal{H} and D denotes the Dirac operator. As a corollary of this, we show that every eigenfunction of the Laplacian or of the Dirac operator on \mathcal{H} extends as a holomorphic eigenfunction of the corresponding holomorphic operator on a certain region of the complexification of \mathcal{H} .

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The case $n = 1$ has been previously studied in [J], [JT], and the Penrose transform investigated fully in [Ts]. In all cases, the Penrose transform is a special case of the \mathcal{P} operator considered in [BKZ]. Our results complement the classic approach of [H1], [H2] and [S], where the integration is on the boundary ('at infinity') of G/K .

Our results can be interpreted in terms of representation theory. In particular, our work fits into the general context of realizing representations of semi-simple Lie groups on cohomology spaces. The underlying Harish-Chandra module of the representation of G on the cohomology space is identified in [W, §9] by means of the Taylor series map. The cases we study are interesting in that the standard vanishing theorems do not apply: we have non-vanishing cohomology in two degrees. Moreover, the cohomology in the 'unexpected' degree does not occur in H^0 , as it does in the more familiar case of the ladder representations.

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2. GEOMETRY

For simplicity we will give most of the discussion in terms of $G = SO_0(2n + 1, 1)$ rather than $\tilde{G} = Spin_0(2n + 1, 1)$. Let G act by its defining representation on \mathbb{R}^{2n+2} (regarded as the space of column vectors) preserving the symmetric bilinear form $\langle \cdot, \cdot \rangle$ with matrix $\text{diag}(1, \dots, 1, -1)$. We use the same notation for the extension of $\langle \cdot, \cdot \rangle$ to a symmetric bilinear form on the complexification \mathbb{C}^{2n+2} . Let \mathfrak{g}_0 denote the Lie algebra of $SO_0(2n + 1, 1)$.

We identify the $(2n + 1)$ -dimensional hyperbolic space \mathcal{H} with one sheet of the hyperboloid of vectors in \mathbb{R}^{2n+2} of length -1 :

$$\mathcal{H} = \{x \in \mathbb{R}^{2n+2} \mid \langle x, x \rangle = -1, x_{2n+2} > 0\}$$

We let $\text{IGr}_n(\mathbb{C}^{2n+2})$ denote the isotropic Grassmannian consisting of all n -dimensional complex subspaces P of \mathbb{C}^{2n+2} with the property that the restriction of $\langle \cdot, \cdot \rangle$ to P is zero. Under G (or \tilde{G}), this Grassmannian is a union of two orbits. The open orbit \mathcal{T} consists of all P with trivial intersection with \mathbb{R}^{2n+2} . We say that $x \in \mathcal{H}$ and $P \in \mathcal{T}$ are *incident* if $\langle x, y \rangle = 0, \forall y \in P$. Setting F to be the subset of $\mathcal{H} \times \mathcal{T}$ defined by incidence, we have the following double fibration:

$$(1) \quad \begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ \mathcal{T} & & \mathcal{H} \end{array}$$

It is easy to check that the fibres of τ are copies of the isotropic Grassmannian $\text{IGr}_n(\mathbb{C}^{2n+1})$ and the fibres of η are diffeomorphic to \mathbb{R} .

2.1. Description as homogeneous spaces. We identify \mathcal{T} with G/L , where L is the stabiliser in G of the isotropic n -dimensional subspace

$$\left\{ (z_1, iz_1, \dots, z_n, iz_n, 0, 0)^T \mid z_1, \dots, z_n \in \mathbb{C} \right\}.$$

Thus, in block form,

$$L = \left\{ \begin{pmatrix} U & 0 \\ 0 & A \end{pmatrix} \mid U \in U(n), A \in SO_0(1, 1) \right\},$$

where we are embedding $U(n)$ in $SO(2n, \mathbb{R})$ by identifying \mathbb{C}^n with \mathbb{R}^{2n} in the usual way. Also, $\mathcal{H} = G/K$, where $K = SO(2n + 1)$ embedded in the top left corner of G . Setting $M = L \cap K \cong U(n)$, we have $F = G/M$.

The above picture can be embedded in the complex holomorphic correspondence

$$(2) \quad \begin{array}{ccc} & \mathbb{C}F & \\ \mu \swarrow & & \searrow \nu \\ \text{IGr}_n(\mathbb{C}^{2n+2}) & & \mathbb{C}\mathcal{H} \end{array}$$

where we define holomorphic hyperbolic space $\mathbb{C}\mathcal{H}$ by

$$\mathbb{C}\mathcal{H} = \{x \in \mathbb{C}\mathbb{P}_{2n+1} \mid \langle x, x \rangle \neq 0\},$$

and the correspondence is defined as before. This correspondence is homogeneous for $G^{\mathbb{C}} = SO(2n + 2, \mathbb{C})$. Keeping the same reference points as above, we identify $\text{IGr}_n(\mathbb{C}^{2n+2})$ with $G^{\mathbb{C}}/Q$, where Q is a certain θ -stable parabolic that is easily computed. We also have $\mathbb{C}\mathcal{H} = G^{\mathbb{C}}/(SO(2n + 1, \mathbb{C}) \times \mathbb{Z}_2)$. Setting

$$R = (SO(2n + 1, \mathbb{C}) \times \mathbb{Z}_2) \cap Q,$$

we have that $\mathbb{C}F = G^{\mathbb{C}}/R$. We identify the Lie algebra of $G^{\mathbb{C}}$ with the complexification \mathfrak{g} of \mathfrak{g}_0 .

3. INVOLUTIVE STRUCTURES AND COHOMOLOGY

For the moment, let us proceed in some generality, letting G denote a Lie group and M a closed subgroup, so that G/M is a homogeneous space. Let \mathfrak{m}_0 denote the Lie algebra of M and let \mathfrak{m} denote its complexification. Let \mathfrak{q} be a complex Lie subalgebra of \mathfrak{g} containing \mathfrak{m} . Associated to \mathfrak{q} there is a distinguished homogeneous formally integrable subbundle $T_{\mathfrak{q}}^{0,1}G/M$ of the complexified tangent bundle of G/M which is associated to the inclusion $\mathfrak{q}/\mathfrak{m} \rightarrow \mathfrak{g}/\mathfrak{m}$. This endows G/M with a G -homogeneous *involutive structure*. (For a general exposition on involutive or formally integrable structures, see [T]. For a complete discussion of their application in the Penrose transform, see [BES]. For a brief discussion, see [D].) We will write $(G/M, \mathfrak{q})$ henceforth for G/M equipped with the homogeneous involutive structure defined by \mathfrak{q} .

Let $\mathcal{E}_{\mathfrak{q}}^{0,k}$ denote the homogeneous vector bundle on G/M induced from the M -module $\bigwedge^k(\mathfrak{q}/\mathfrak{m})^*$. When $k = 0$, $\mathcal{E}_{\mathfrak{q}}^{0,0} = \mathcal{E}$ is the trivial complex line bundle. Then one has a complex

$$(3) \quad \Gamma(G/M, \mathcal{E}) \xrightarrow{\bar{\partial}} \Gamma(G/M, \mathcal{E}_{\mathfrak{q}}^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(G/M, \mathcal{E}_{\mathfrak{q}}^{0,2}) \xrightarrow{\bar{\partial}} \dots$$

which arises by quotienting the (complexified) de Rham complex by the differential ideal generated by the annihilator of $T_{\mathfrak{q}}^{0,1}G/M$.

If V is a complex (M, \mathfrak{q}) -module, then we say that the associated homogeneous bundle \underline{V} is *compatible* with the involutive structure, and in this case one obtains a complex

$$(4) \quad \Gamma(G/M, \mathcal{E}(V)) \rightarrow \Gamma(G/M, \mathcal{E}_{\mathfrak{q}}^{0,1}(V)) \rightarrow \Gamma(G/M, \mathcal{E}_{\mathfrak{q}}^{0,2}(V)) \rightarrow \dots,$$

where we write $\mathcal{E}_{\mathfrak{q}}^{0,k}(V)$ for $\mathcal{E}_{\mathfrak{q}}^{0,k} \otimes \underline{V}$. We will denote the cohomology of this complex by $H_{\mathfrak{q}}^*(G/M, V)$.

To return to the particular spaces we have been discussing, we will be concerned with several instances of this construction. The first of these is for $(G/L, \mathfrak{q})$, where the involutive structure simply defines the complex structure on G/L arising from its inclusion in $G^{\mathbb{C}}/Q$; the compatible bundles are precisely the holomorphic ones and the cohomology is the usual Dolbeault cohomology of holomorphic vector bundles. The second instance is for $(G/M, \mathfrak{q})$, where the symbols have the meaning of previous sections. Finally, we can consider the case of $(G/M, \mathfrak{r})$. In this case, it is easy to check that the sequence (3) is the Dolbeault resolution along the fibres of τ . If V is an (M, \mathfrak{r}) -module, then the induced vector bundle \underline{V} is holomorphic when restricted to fibres of τ , and the cohomology $H_{\mathfrak{r}}^k(G/M, V)$ is the fibrewise Dolbeault cohomology. This cohomology is isomorphic to smooth sections over G/K of a complex vector bundle, which we will denote by $\tau_*^k V$.

4. HOMOGENEOUS BUNDLES

A general element of the Lie algebra \mathfrak{l}_0 is of the form

$$(5) \quad \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}, \quad u \in \mathfrak{u}(n), a \in \mathbb{R}.$$

For $k \in \mathbb{Z}, \lambda \in \mathbb{C}$, the expression $-\frac{k}{2}(\text{Tr } u) - \lambda a$ is the differential of a character of \tilde{L} , where $G/L = \tilde{G}/\tilde{L}$. (Here we are taking the trace of u as an $n \times n$ complex matrix, not as a $2n \times 2n$ real matrix.) These all admit a unique extension to $(\tilde{L}, \mathfrak{q})$ -modules, and hence define holomorphic \tilde{G} -homogeneous line bundles over \mathcal{T} , which we will denote by $\mathcal{O}(k, \lambda)$.

The typical element of \mathfrak{m}_0 is as in (5) above, but with $a = 0$. For $k \in \mathbb{Z}$, the expression $-\frac{k}{2}(\text{Tr } u)$ is the differential of a character of \tilde{M} . These extend uniquely to $(\tilde{M}, \mathfrak{r})$ -modules, and hence the associated homogeneous line bundle on G/M , which we denote by $\mathcal{O}(k)$, is compatible with the involutive structure defined by \mathfrak{r} .

Also we have the representation of \mathfrak{m}_0 which is the defining representation of u as a complex $n \times n$ matrix. This is also the differentiated form of a representation of M , and it extends uniquely to \mathfrak{r} . We denote this (M, \mathfrak{r}) -module by V . The corresponding vector bundle \underline{V} is thus compatible with the involutive structure defined by \mathfrak{r} .

On G/K we will need to consider the 2^n complex-dimensional vector bundle $\underline{\mathbf{S}}$ of spinors, associated to the spin representation \mathbf{S} of $K = SO(2n+1)$. It is interesting to note that the fibre bundle $F \rightarrow \mathcal{H}$ of (1) is precisely the projectivisation of the bundle of pure spinors.

5. THE PENROSE TRANSFORM

Let \underline{E} be a holomorphic L -homogeneous bundle on \mathcal{T} associated to the (L, \mathfrak{q}) -module E . Since the fibres of η are contractible, it is trivial to show that pull-back provides an isomorphism

$$H_{\mathfrak{q}}^k(G/L, E) \xrightarrow{\cong} H_{\mathfrak{q}}^k(G/M, E).$$

Let Ω_{η}^1 denote the homogeneous bundle corresponding to the (M, \mathfrak{r}) -module $(\mathfrak{q}/\mathfrak{r})^*$. Then the short exact sequence of (M, \mathfrak{r}) -modules

$$0 \rightarrow (\mathfrak{q}/\mathfrak{r})^* \rightarrow (\mathfrak{q}/\mathfrak{m})^* \rightarrow (\mathfrak{r}/\mathfrak{m})^* \rightarrow 0$$

corresponds to the short exact sequence of vector bundles on G/M

$$0 \rightarrow \Omega_\eta^1 \rightarrow \mathcal{E}_q^{0,1} \rightarrow \mathcal{E}_\tau^{0,1} \rightarrow 0.$$

This induces a filtration on the complex $\Gamma(G/M, \mathcal{E}_q^{0,k}(E))$, which then computes $H_q^k(G/M, E)$ and hence $H_q^k(G/L, E)$. In the associated spectral sequence, we have

$$E_1^{p,q} = H_\tau^q(G/M, \Omega_\eta^p \otimes \underline{E}),$$

where $\Omega_\eta^p = \bigwedge^p \Omega_\eta^1$. Identifying this cohomology with sections of vector bundles over G/K as at the end of §3, we obtain:

Proposition 5.1. *There is a spectral sequence with*

$$E_1^{p,q} = \Gamma(G/K, \tau_*^q(\Omega_\eta^p \otimes \underline{E})) \implies H_q^{p+q}(G/L, E).$$

This spectral sequence is what we call the *Penrose transform*. In order to obtain concrete results from it, one needs to identify the vector bundles over G/K which are appearing.

6. COMPUTATION OF τ_*^k

The fibre of τ over the identity coset in G/K can be identified with K/M , which is the projectivisation of the space of pure spinors for $SO(2n + 1)$. This comes equipped with a natural K -invariant complex structure, which in the language of homogeneous involutive structures is given by $(K/M, \mathfrak{r})$. If \underline{E} is a homogeneous vector bundle over G/M associated to the (M, \mathfrak{r}) -module E , then $\tau_*^k E$ is the vector bundle over G/K associated to the representation of K on $H_\tau^k(K/M, E|_{(M, \mathfrak{r})})$. This can be computed by the Bott-Borel-Weil Theorem, and we obtain:

Lemma 6.1. *Let $-2n - 1 \leq p \leq -1$. Then $H^q(K/M, \bigwedge^k(\underline{V}^*) \otimes \mathcal{O}(p))$ is identically zero, with the following exceptions:*

p	k	q	$H_\tau^q(K/M, \bigwedge^k(\underline{V}^*) \otimes \mathcal{O}(p))$
$-2n - 1$	0	$n(n + 1)/2$	S
$-2n$	0	$n(n + 1)/2$	\mathbb{C}
$-2n$	1	$n(n + 1)/2 - 1$	\mathbb{C}
-2	$n - 1$	1	\mathbb{C}
-2	n	0	\mathbb{C}
-1	n	0	S

where **S** denotes the spin representation of K and \mathbb{C} the trivial one-dimensional representation.

Lemma 6.2. *Let $-2n - 1 \leq p \leq -1$. Then the cohomology $H^q(K/M, \Omega_\eta^k \otimes \mathcal{O}(p))$ is identically zero, with the following exceptions:*

p	k	q	$H_\tau^q(K/M, \Omega_\eta^k \otimes \mathcal{O}(p))$
$-2n - 1$	0	$n(n + 1)/2$	S
$-2n - 1$	1	$n(n + 1)/2$	S
$-2n$	0	$n(n + 1)/2$	\mathbb{C}
$-2n$	2	$n(n + 1)/2 - 1$	\mathbb{C}
-2	$n - 1$	1	\mathbb{C}
-2	$n + 1$	0	\mathbb{C}
-1	n	0	S
-1	$n + 1$	0	S

Proof. It is elementary to check that there is a short exact sequence of (M, \mathfrak{r}) -modules which leads to a short exact sequence of compatible vector bundles

$$(6) \quad 0 \rightarrow \mathcal{O} \rightarrow \Omega_\eta^1 \rightarrow \underline{V}^* \rightarrow 0.$$

Taking exterior powers of this and tensoring through by $\mathcal{O}(p)$, we obtain

$$(7) \quad 0 \rightarrow \wedge^{k-1} \underline{V}^*(p) \rightarrow \Omega_\eta^k(p) \rightarrow \wedge^k \underline{V}^*(p) \rightarrow 0$$

for $2 \leq k \leq n$. Also, $\Omega_\eta^{n+1}(p) \cong \mathcal{O}(p+2)$. We now apply the results of the previous lemma to the long exact sequences of cohomology coming from (7). This immediately yields all except the following two cases. In the case of $H_\tau^1(K/M, \Omega_\eta^n(-2))$ the long exact sequence contains the segment

$$\begin{array}{ccccccc} H_\tau^0(K/M, \mathcal{O}) & \xrightarrow{\alpha} & H_\tau^1(K/M, \wedge^{n-1} \underline{V}^*(-2)) & \rightarrow & H_\tau^1(K/M, \Omega_\eta^n(-2)) & \rightarrow & 0 \\ \parallel & & \parallel & & & & \\ \mathbb{C} & & \mathbb{C} & & & & \end{array}$$

Here, $\alpha(1)$ is the extension class of (6), which it is easy to check is non-trivial. (Cf. [H, Ch III § 6, Ex. 1].) Hence $H_\tau^1(K/M, \Omega_\eta^n(-2)) = 0$. By Serre duality,

$$H_\tau^{\frac{1}{2}n(n+1)-1}(K/M, \Omega_\eta^1(-2n)) = 0,$$

which is the other case for which the long exact sequences do not immediately give the result. □

7. RESULTS

Theorem 7.1. *The Penrose transform gives the following isomorphisms.*

- (1) *Let $-2n < k < -2$. Then $H^p(\mathcal{T}, \mathcal{O}(k, \lambda)) = 0$ for all p, λ .*
- (2) *The cohomology $H^{n(n+1)/2}(\mathcal{T}, \mathcal{O}(-2n, \lambda))$ is isomorphic to the kernel of the operator*

$$\Delta - (\lambda^2 - n^2)$$

acting on functions on \mathcal{H} . The cohomology in degree $\frac{n(n+1)}{2} + 1$ is the cokernel of this operator, and all other cohomology vanishes.

- (3) *The cohomology $H^n(\mathcal{T}, \mathcal{O}(-2, \lambda))$ is isomorphic to the kernel of the operator*

$$\Delta - (\lambda^2 - n^2)$$

acting on functions on \mathcal{H} . The cohomology in degree $(n + 1)$ is the cokernel of this operator, and all other cohomology vanishes.

- (4) *The cohomology $H^{n(n+1)/2}(\mathcal{T}, \mathcal{O}(-2n - 1, \lambda))$ is isomorphic to the kernel of the operator*

$$D + i\lambda,$$

where D denotes the Dirac operator on smooth sections of the spin bundle over \mathcal{H} . The cohomology in degree $\frac{n(n+1)}{2} + 1$ is the cokernel of this operator, and all other cohomology vanishes.

- (5) *The cohomology $H^n(\mathcal{T}, \mathcal{O}(-1, \lambda))$ is isomorphic to the kernel of the operator*

$$D - i\lambda,$$

where D denotes the Dirac operator on spinors on \mathcal{H} . The cohomology in degree $(n + 1)$ is the cokernel of this operator, and all other cohomology vanishes.

Proof. We apply the spectral sequence of Proposition 5.1, using the results of Lemma 6.2 to identify the terms. In case (1), all terms in the E_1 level are zero. In case (2), all terms are zero except

$$E_1^{0,n(n+1)/2} = E_1^{2,n(n+1)/2-1} = \Gamma(\mathcal{H}, \mathcal{E}).$$

Thus the spectral sequence converges after the E_2 level, where the differential is a second-order differential operator between functions on G/K . By group invariance, this is some multiple of the Laplace operator plus some constant. (Cf. [H2, Ch. II].) The constant is identified by comparing the actions of the Casimir operator $\Omega_{\mathfrak{g}}$ for \mathfrak{g} on $H^p(\mathcal{T}, \mathcal{O}(k, \lambda))$ and on functions on \mathcal{H} . The action of $\Omega_{\mathfrak{g}}$ on the cohomology space is determined from the central character to be multiplication by

$$(8) \quad n \left(\frac{k}{2}\right)^2 + n(n+1) \left(\frac{k}{2}\right) + \lambda^2.$$

The Casimir operator acts on functions on \mathcal{H} as the Laplacian. Letting $k = -2n$ computes the eigenvalue. Case (3) is very similar.

In cases (4) and (5), the spectral sequence converges at the E_1 level with first-order operators between spinors on \mathcal{H} . Now the eigenvalue is computed by comparing the actions of $\Omega_{\mathfrak{g}}$ on spinor fields on \mathcal{H} and on $H^{n(n+1)/2}(\mathcal{T}, \mathcal{O}(-2n-1, \lambda))$ for case (4) or $H^n(\mathcal{T}, \mathcal{O}(-1, \lambda))$ for case (5).

The argument in [AS, Appendix] for the twisted Dirac operator on the symmetric space G/K , where $\text{rank}(G) = \text{rank}(K)$, extends to the case of arbitrary symmetric spaces to show that the spinor Laplacian on sections of spinors twisted by a vector bundle \underline{V}_μ acts by

$$D^2 = -r(\Omega_{\mathfrak{g}}) \otimes 1 \otimes 1 + 1 \otimes \tau_\mu(\Omega_{\mathfrak{k}}) \otimes 1 + 1 \otimes 1 \otimes s(\Omega_{\mathfrak{k}}).$$

Here, $\Omega_{\mathfrak{k}}$ is the Casimir operator for \mathfrak{k} , \underline{V}_μ is the homogeneous vector bundle on G/K corresponding to (τ_μ, V_μ) , the irreducible representation of K with highest weight μ , and $s(\cdot)$ is the action of $\mathcal{U}(\mathfrak{k})$ on \mathbf{S} . Also, we have written $C^\infty(G/K, \underline{V}_\mu \otimes \underline{\mathbf{S}})$ as $(C^\infty(G) \otimes V_\mu \otimes \mathbf{S})_K$, the subscript K denoting the K -invariants. In the case at hand, V_μ is the trivial representation, hence $\tau_\mu(\Omega_{\mathfrak{k}}) = 1$. Also in our case, we compute that $\Omega_{\mathfrak{k}}$ acts on spinors as multiplication by $\frac{2n^2+n}{4}$. Thus, we have $D^2 = -\Omega_{\mathfrak{g}} - \frac{2n^2+n}{4}$. Recalling from (8) the eigenvalue of the action of $\Omega_{\mathfrak{g}}$ for the given central character and letting $k = -2n - 1$ for case (4) and $k = -1$ for case (5), we obtain that D^2 acts by $-\lambda^2$ on the image of the transform. \square

Corollary 7.2. *Let $\hat{\mathcal{H}}$ denote the open subset of complex holomorphic hyperbolic space $\mathbb{C}\mathcal{H}$ consisting of all points x such that the corresponding compact complex submanifold of $\text{IGr}_n(\mathbb{C}^{2n+2})$ according to the correspondence (2) is contained in \mathcal{T} . Then every smooth eigenfunction of the Dirac or Laplace operator on \mathcal{H} is the restriction of a holomorphic eigenfunction of the corresponding holomorphic operator on $\hat{\mathcal{H}}$.*

Proof. The holomorphic Penrose transform provides a map from cohomology on \mathcal{T} to such holomorphic fields on $\hat{\mathcal{H}}$, and on restriction to \mathcal{H} it agrees with our Penrose transform. \square

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