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THE STABILITY RADIUS OF A QUASI-FREDHOLM OPERATOR

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Abstract. We extend the technique used by Kordula and Müller to show that the stability radius of a quasi-Fredholm operator T is the limit of $\gamma(T^n)^{1/n}$ as $n \to \infty$. If 0 is an isolated point of the Apostol spectrum $\sigma_\gamma(T)$, then the above limit is non-zero if and only if T is quasi-Fredholm.

Let $L(X)$ be the set of all bounded linear operators on a complex Banach space X. For any $T \in L(X)$, we denote the null space and range of T by $N(T)$ and $R(T)$ respectively. The *Apostol spectrum* of T is defined to be the set

$$\sigma_\gamma(T) = \{\, \mu \in \mathbb{C} : \lim_{\lambda \to \mu} \gamma(T - \lambda I) = 0 \,\}, \tag{1}$$

where $\gamma(T)$ is the reduced minimum modulus of T, that is,

$$\gamma(T) = \begin{cases} \inf\{\, \|Tx\| : x \in X, d(x, N(T)) = 1 \,\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases}$$

The Apostol spectrum was first defined in this form by Apostol in [1] for operators on a Hilbert space. Its complement in $\mathbb{C}$ is usually called the *semi-regular region* of T and is denoted by $\rho_\gamma(T)$. T is *semi-regular* (or s-regular) if $0 \in \rho_\gamma(T)$. Properties of the Apostol spectrum for operators on a Banach space can be found in [10, 11]. The *stability radius* of T is defined as the distance

$$\delta(T) = d(0, \sigma_\gamma(T) \setminus \{0\}). \tag{2}$$

It is the radius of the largest punctured open disc centred at 0 in which $T - \lambda I$ is semi-regular. When T is semi-regular, it is shown in [5] that

$$\delta(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}. \tag{3}$$

In the special case when T is bounded below or surjective, $\delta(T)$ is also the distance from 0 to the approximate point spectrum and to the surjectivity spectrum respectively. When 0 is an isolated point of $\sigma_\gamma(T)$, the formula (3) still applies to certain classes of operators. It includes the cases when T is Fredholm [3], semi-Fredholm [14], essentially s-regular [5], or chain finite (T has finite ascent and descent) [2].

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An operator T is *regular* (or of *Saphar* type) if it is both relatively regular and semi-regular. The stability problem concerning *regular* operators is studied in [13].

Definition. An operator $T \in L(X)$ has *topological uniform descent* d (where d is a nonnegative integer) if $N(T^n)+R(T) = N(T^d)+R(T)$ is a closed subspace for all $n \geq d$ (Grabiner [4]). An operator T is *quasi-Fredholm* if T has topological uniform descent d for some integer d and $R(T^n)$ is closed for all $n \geq d$.

Quasi-Fredholm operators were first defined on a Hilbert space by Labrousse [6] and on a Banach space by Mbekhta and Müller [9]. The definition used here is different from but equivalent to the one given in [9]. The class of quasi-Fredholm operators is well researched for the Hilbert space case [6, 7]. In [6], it was shown that an operator on a Hilbert space is quasi-Fredholm if and only if it has a Kato decomposition. A characterization of quasi-Fredholm operators in a Banach space is examined in [12]. Two pertinent properties of quasi-Fredholm operators are proved in Theorem 7 and Corollary 15 below.

In the present paper, the stability radius problem of a quasi-Fredholm operator is examined. It was shown in [1, Prop. 3.3] that the stability radius formula (3) holds for a Hilbert space operator T if only and if T has a Kato decomposition described in [6], which is of course equivalent to T being quasi-Fredholm. The main aim of this paper is to extend the result to Banach space operators. It turns out that the technique used by Kordula and Müller in [5] can be extended to solve the stability radius problem for quasi-Fredholm operators. The main result of this paper is the following theorem, which is a consequence of Theorem 10 and Corollary 15.

Theorem. *If T is a quasi-Fredholm operator, then the stability radius of T is equal to $\lim_{n\to\infty} \gamma(T^n)^{1/n}$. Moreover, if 0 is an isolated point of the Apostol spectrum of an arbitrary operator T, then $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ always exists. This limit is non-zero if and only if T is quasi-Fredholm.*

If M, N are closed T-invariant subspaces with $N \subseteq M$, then we denote the map induced by T on the quotient M/N by $T_{M/N}$. More precisely, $T_{M/N}$ is the map $x+N \mapsto Tx+N$. We also denote the restriction of T to M by T_M. The ascent and descent of T will be denoted by $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ respectively. The hyperkernel $\bigcup_{n=1}^{\infty} N(T^n)$ and hyperrange $\bigcap_{n=1}^{\infty} R(T^n)$ of T are denoted by $N(T^\infty)$ and $R(T^\infty)$ respectively. We first prove some properties of the reduced minimum modulus.

Lemma 1. *Let M be a closed T-invariant subspace.*

(i) *If $T_{X/M}$ is injective, then $\gamma(T_M) \geq \gamma(T)$.*
(ii) *If T_M has dense range, then $\gamma(T_{X/M}) \geq \gamma(T)$.*

Proof. (i) The hypothesis shows that $T^{-1}M \subseteq M$. In particular, $N(T) \subseteq M$. Hence $N(T_M) = N(T) \cap M = N(T)$. It is clear from the definition of the minimum modulus that $\gamma(T_M) \geq \gamma(T)$.

(ii) If $\gamma(T_{X/M}) = \infty$, the result is trivial. If $\gamma > \gamma(T_{X/M})$, we can find $x \in X$ such that $d(x+M, N(T_{X/M})) = 1$ and $\|T_{X/M}(x+M)\| < \gamma$. So

$$d(x, T^{-1}M) = 1 \quad \text{and} \quad d(Tx, M) < \gamma.$$

Since TM is dense in M, we can find $m \in M$ such that $\|T(x+m)\| < \gamma$. Note that $N(T) \subseteq T^{-1}M$. As $TM \subseteq M$, we also have $M \subseteq T^{-1}M$. Hence

$$\gamma(T) \leq \frac{\|T(x+m)\|}{d(x+m, N(T))} \leq \frac{\|T(x+m)\|}{d(x+m, T^{-1}M)} = \frac{\|T(x+m)\|}{d(x, T^{-1}M)} < \gamma.$$

Considering all $\gamma > \gamma(T_{X/M})$, we conclude that $\gamma(T_{X/M}) \geq \gamma(T)$. $\square$

Lemma 2. *Suppose* $\operatorname{asc}(T) = d$, *i.e.* $N(T^d) = N(T^n)$ *for all* $n \geq d$. *Let* $\widehat{T}$ *be the map induced by* T *on the quotient* $X/N(T^d)$*; then*

$$\liminf_{n\to\infty} \gamma(T^n)^{1/n} = \liminf_{n\to\infty} \gamma(\widehat{T}^n)^{1/n}, \quad \limsup_{n\to\infty} \gamma(T^n)^{1/n} = \limsup_{n\to\infty} \gamma(\widehat{T}^n)^{1/n}. \tag{4}$$

Moreover, $\limsup_{n\to\infty} \gamma(T^n)^{1/n} > 0 \iff \widehat{T}$ *is bounded below* $\iff \lim_{n\to\infty} \gamma(T^n)^{1/n} > 0$.

Proof. Let us first assume that $T^d \neq 0$. Since $\operatorname{asc}(T) = d$, we have $T^{-1}N(T^d) = N(T^d)$ and $\widehat{T}$ is injective. Taking any $x \in X$, we have

$$\|T^n x\| \geq d(T^n x, N(T^d)) \geq \|T^d\|^{-1}\|T^{n+d}x\|. \tag{5}$$

The first inequality is obvious. The second follows from the relation

$$\|T^{n+d}x\| = \|T^d(T^n x + z)\| \leq \|T^d\|\|T^n x + z\|,$$

true for all $z \in N(T^d)$. Write $x + N(T^d)$ as $\widehat{x}$; then for $n \geq d$, we have

$$\|\widehat{T}^n \widehat{x}\| = d(T^n x, N(T^d)), \quad \|\widehat{x}\| = d(x, N(T^d)) = d(x, N(T^n)) = d(x, N(T^{n+d})).$$

We deduce from (5) that $\gamma(T^n) \geq \gamma(\widehat{T}^n) \geq \|T^d\|^{-1}\gamma(T^{n+d})$ by taking infima over all x with $\|\widehat{x}\| = 1$; (4) then follows by taking limits. When $T^d = 0$, $\widehat{T} = 0$ and all limits in (4) are infinite. To establish the last statement, we proceed as follows:

(a) If $\limsup_{n\to\infty} \gamma(T^n)^{1/n} > 0$, then it follows from (4) that $\gamma(\widehat{T}^k) > 0$ for some k. Therefore $R(\widehat{T}^k)$ is a closed subspace. Since $\widehat{T}$ is injective, so is $\widehat{T}^k$. Thus $\widehat{T}^k$ is injective and has closed range. So $\widehat{T}^k$ is bounded below. Hence the Banach space adjoints of $\widehat{T}^k$ and $\widehat{T}$ are surjective. So $\widehat{T}$ is bounded below.

(b) If $\widehat{T}$ is bounded below, then the limit $\lim_{n\to\infty} \gamma(\widehat{T}^n)^{1/n}$ exists and is positive [8]. By (4), the same is true for $\lim_{n\to\infty} \gamma(T^n)^{1/n}$.

(c) If $\lim_{n\to\infty} \gamma(T^n)^{1/n} > 0$, then obviously $\limsup_{n\to\infty} \gamma(T^n)^{1/n} > 0$. $\square$

By applying the lemma to T^* instead of T and using Banach space duality, one can readily verify the following lemma.

Lemma 3. *Suppose* $\overline{R(T^n)} = \overline{R(T^d)}$ *for all* $n \geq d$. *Let* $\widetilde{T}$ *be the restriction of* T *to* $\overline{R(T^d)}$*; then*

$$\liminf_{n\to\infty} \gamma(T^n)^{1/n} = \liminf_{n\to\infty} \gamma(\widetilde{T}^n)^{1/n},$$
$$\limsup_{n\to\infty} \gamma(T^n)^{1/n} = \limsup_{n\to\infty} \gamma(\widetilde{T}^n)^{1/n}.$$

Moreover, $\limsup_{n\to\infty} \gamma(T^n)^{1/n} > 0 \iff \widetilde{T}$ *is surjective* $\iff \lim_{n\to\infty} \gamma(T^n)^{1/n} > 0$.

The following lemma is a refinement of Kordula and Müller [5, Lemma 2]. The technique used in the proof is adapted from that paper.

Lemma 4. *Let* $N \subseteq M$ *be closed* T*-invariant subspaces of* X *such that* $T_{X/M}$ *is bounded below,* T_N *is surjective and* $T^d M \subseteq N$. *Then:*

(i) $N = R(T^n) \cap M = T^n M$ *for each* $n \geq d$.
(ii) $M = N(T^n) + N = T^{-n}N$ *for each* $n \geq d$.

(iii) $\operatorname{des}(T_M) \le d$, $R(T_M^d) = N$ *and*

$$\lim_{n\to\infty} \gamma(T_M^n)^{1/n} = \lim_{n\to\infty} \gamma(T_N^n)^{1/n}.$$

(iv) $\operatorname{asc}(T_{X/N}) \le d$, $N(T_{X/N}^d) = M/N$ *and*

$$\lim_{n\to\infty} \gamma(T_{X/N}^n)^{1/n} = \lim_{n\to\infty} \gamma(T_{X/M}^n)^{1/n}.$$

(v) $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \min\{ \lim_{n\to\infty} \gamma(T_{X/M}^n)^{1/n}, \lim_{n\to\infty} \gamma(T_N^n)^{1/n} \}$.

Proof. Consider any integer $n \ge d$.

(i) Since $T_{X/M}$ is injective, we have $T^{-1}M = M$. So $T^nM = T^nT^{-n}M = M \cap R(T^n)$. From the fact that T_N is surjective, $N \subseteq M$ and $T^dM \subseteq N$, we have the inclusions $N \subseteq T^nN \subseteq T^nM \subseteq N$. This establishes (i).

(ii) Since T_N is surjective, we have $T^{-n}N = T^{-n}T^nN = N(T^n) + N$. From the fact that $T^nM \subseteq N$, $N \subseteq M$ and $T_{X/M}$ is injective, we have the inclusions $M \subseteq T^{-n}N \subseteq T^{-n}M \subseteq M$. This shows (ii).

(iii) It follows from (i) that $\operatorname{des}(T_M) \le d$, $R(T_M^d) = N$. Using the notation in Lemma 3, we have $(T_M)^\sim = T_N$, a surjective operator by hypothesis. The rest of (iii) follows from an application of Lemma 3 to the operator T_M.

(iv) It follows from (ii) that $N(T_{X/N}^n) = T^{-n}N/N = M/N$ for all $n \ge d$. So $\operatorname{asc}(T_{X/N}) \le d$ and $N(T_{X/N}^d) = M/N$. Using the notation in Lemma 2, we can identify $(T_{X/N})^\wedge$ with $T_{X/M}$, which is bounded below. The rest of (iv) follows from an application of Lemma 2 to the operator $T_{X/N}$.

(v) From (iii), (iv) and Lemma 1,

$$\limsup_{n\to\infty} \gamma(T^n)^{1/n} \le \min\{ \lim_{n\to\infty} \gamma(T_{X/M}^n)^{1/n}, \lim_{n\to\infty} \gamma(T_N^n)^{1/n} \}.$$

To prove the reverse inequality, we adopt the approach of [5, Lemma 2]. We assume that $M \ne X$ and $N \ne 0$; otherwise, (v) follows from (iii) and (iv) directly. This means that T_N, $T_{X/M}$ are non-zero operators which are either surjective or bounded below. Hence, both $\gamma(T_N^n)$, $\gamma(T_{X/M}^n)$ are finite and positive for all n. Since (v) holds if and only if it holds for some non-zero multiple of T, we can also assume without loss of generality that $\|T\| = 1$. For each $i \ge d$, let γ_i^{-1} be the maximum of $\gamma(T_N^i)^{-1}$ and $\gamma(T_{X/M}^i)^{-1}$. We also let $t > 1$ and $n \ge d$, and we let x be an arbitrary unit vector in $R(T^n)$. For each $i = d, \dots, n$, it is possible to pick $x_i \in T^{-i}[x + M]$ such that $\|x_i\| \le td(x_i, M)$. Since $d(x_i, M) \le \gamma(T_{X/M}^i)^{-1} d(x, M)$,

$$\|x_i\| \le t\gamma(T_{X/M}^i)^{-1} \|x\| \le t\gamma_i^{-1}.$$

Let $m_i = Tx_{i+1} - x_i$ for $i = d, \dots, n-1$; then

$$\|m_i\| \le \|Tx_{i+1}\| + \|x_i\| \le \|x_{i+1}\| + \|x_i\| \le t(\gamma_{i+1}^{-1} + \gamma_i^{-1}).$$

and $\sum_{i=d}^{n-1} T^i m_i + T^d x_d - x = T^n x_n - x$. It is clear from the definition of x_i that $T^i m_i \in M$. So $m_i \in T^{-i}M \subseteq M$. If $i \ge d$, then $T^d m_i \in R(T^d) \cap M = N$ by (i). As T_N is surjective, there exists $u_i \in N$ such that $T^d m_i = T^{n-i+d} u_i$ and $\|u_i\| \le td(u_i, N(T^{n-i+d}))$. Therefore,

$$\|u_i\| \le t\gamma(T_N^{n-i+d})^{-1} \|T^d m_i\| \le t\gamma_{n-i+d}^{-1} \|m_i\| \le t^2 \gamma_{n-i+d}^{-1} (\gamma_{i+1}^{-1} + \gamma_i^{-1}).$$

It is easy to verify that $T^i m_i = T^n u_i$. Let $m = T^d x_d - x$. Since $x \in R(T^n)$, we have $m \in R(T^d) \cap M = N$. We can pick $u \in N$ with $m = T^n u$, $\|u\| \le t d(u, N(T_N^n))$,

$$\|u\| \le t\gamma(T_N^n)^{-1}\|m\| \le t\gamma_n^{-1}\|m\| \le t^2\gamma_n^{-1}(\gamma_d^{-1}+1).$$

Let $z = x_n - \sum_{i=d}^{n-1} u_i - u$. We now have

$$T^n z = T^n x_n - \sum_{i=d}^{n-1} T^i m_i - (T^d x_d - x) = x,$$

$$d(z, N(T^n)) \le \|z\| \le C\Big[\gamma_n^{-1} + \sum_{i=d}^{n-1} \gamma_{n-i+d}^{-1}(\gamma_{i+1}^{-1} + \gamma_i^{-1}) + \gamma_n^{-1}(\gamma_d^{-1}+1)\Big],$$

where C is the constant $\max\{t, t^2\}$, which is independent of n and x. Since x is an arbitrary unit vector in $R(T^n)$, we have

$$\gamma(T^n)^{-1} \le C\Big[\gamma_n^{-1} + \sum_{i=d}^{n-1} \gamma_{n-i+d}^{-1}(\gamma_{i+1}^{-1} + \gamma_i^{-1}) + \gamma_n^{-1}(\gamma_d^{-1}+1)\Big].$$

Suppose $0 < \gamma < \min\{\lim_{n\to\infty} \gamma(T_{X/M}^n)^{1/n}, \lim_{n\to\infty} \gamma(T_N^n)^{1/n}\}$; then for large enough i, say $i \ge n_0$, we have $\gamma^{-i} \ge \gamma_i^{-1}$. Let $K = 1 + \max_{d \le i \le n_0} \gamma^i \gamma_i^{-1}$; then $\gamma_i^{-1} \le K\gamma^{-i}$ for all i. It is a routine calculation that

$$\begin{aligned}\gamma(T^n)^{-1} &\le CK^2\Big[\gamma^{-n} + \sum_{i=d}^{n-1}(\gamma^{-n-d-1} + \gamma^{-n-d}) + \gamma^{-n-d} + \gamma^{-n}\Big] \\ &\le \gamma^{-n}\, CK^2(3 + 2n - 2d)\max\{1, \gamma^{-d}, \gamma^{-d-1}\}.\end{aligned}$$

Taking limits, we have $\liminf_{n\to\infty} \gamma(T^n)^{1/n} \ge \gamma$. By considering all possible γ, we have

$$\liminf_{n\to\infty} \gamma(T^n)^{1/n} \ge \min\{\lim_{n\to\infty} \gamma(T_{X/M}^n)^{1/n}, \lim_{n\to\infty} \gamma(T_N^n)^{1/n}\}. \quad \square$$

Using the lemma, the stability radius formula can be proved via the Apostol representation for quasi-Fredholm operator [12]. For the sake of completeness, we give an independent proof of the result.

Lemma 5 ([5, Lemma 1])**.** *T is s-regular if and only if there exists a closed subspace M with $TM = M$ and $T_{X/M}$ bounded below. We may choose M to be $R(T^\infty)$.*

Lemma 6. *Let T be quasi-Fredholm; then $\delta(T) > 0$. If Ω is the component of $\rho_\gamma(T)$ containing $\{\lambda : 0 < |\lambda| < \delta(T)\}$ and d is the uniform descent of T, then*

$$R[(T-\lambda I)^\infty] = R(T^\infty) + N(T^\infty) = R(T^\infty) + N(T^d)$$

for all $\lambda \in \Omega$.

Proof. See [4, Theorem 4.7], and note that $T - \lambda I$ has closed range and uniform descent for $n \ge 0$ if and only if $\lambda \in \rho_\gamma(T)$ [10, Corollaire 4.2 (iii)]. $\square$

Theorem 7. *T is a quasi-Fredholm operator if and only if there exist closed T-invariant subspaces M, N with $N \subseteq M$ such that $T_{X/M}$ is bounded below, T_N is surjective and $T^d M \subseteq N$ for some nonnegative integer d. We may take $N = R(T^\infty)$ and $M = N(T^d) + R(T^\infty)$, where d is the uniform descent of T.*

Proof. Suppose there are subspaces M, N with the required properties; then the requirements for Lemma 4 are satisfied. In particular, $N = R(T^n) \cap M$ and $M = N(T^n) + N$ for each $n \geq d$. As $T_{X/M}$ is bounded below, $R(T_{X/M})$ is closed. So is $R(T) + M$. Since

$$R(T) + M = R(T) + N(T^n) + N = R(T) + N(T^n) \quad \text{for all } n \geq d,$$

T has topological uniform descent for $n \geq d$.

It remains to show that $R(T^n)$ is closed for all $n \geq d$. For any $n \geq d$,

$$M + R(T^n) = N(T^d) + N + R(T^n) = N(T^d) + R(T^n),$$

which is a closed subspace by [4, Theorem 3.2]. Clearly $M \cap R(T^n) = N$ is also a closed subspace. Now both M and $R(T^n)$ are paracomplete [6, Prop. 2.1.4]. Using the Neubauer Lemma [6, Prop. 2.1.2], we deduce that $R(T^n)$ is closed. Hence T is quasi-Fredholm.

Conversely, assume T is quasi-Fredholm with uniform descent d. Take $N = R(T^\infty)$ and $M = N(T^d) + R(T^\infty)$. It is clear that M and N are T-invariant subspaces. It is also clear that $N \subseteq M$ and $T^d M \subseteq N$. Since $R(T^n)$ is closed for $n \geq d$ and $N = R(T^\infty)$, N is closed. Moreover, we have $TN = N$ [4, Theorem 3.4]. So T_N is surjective and $T^{-n}N = N + N(T^n)$ for all n. By Lemma 6, we have $M = N + N(T^n) = T^{-n}N$ for all $n \geq d$. It follows that M is closed and $T^{-1}M = M$. Hence, $T_{X/M}$ is injective. Also,

$$R(T) + M = R(T) + N + N(T^d) = R(T) + N(T^d),$$

which is a closed subspace by the definition of topological uniform descent. We conclude that $R(T_{X/M})$ is closed and hence $T_{X/M}$ is bounded below. □

Theorem 8. *If* $\operatorname{des}(T) = d$ *and* $R(T^d)$ *is closed, then the stability radius of* T *is given by* $\delta(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}$.

Remark. We readily deduce from [4, Corollary 4.8 (c)] that $T - \lambda I$ is surjective for every $\lambda \in \Omega$, where Ω is the component of $\rho_\gamma(T)$ defined in Lemma 6.

Proof. We assume that $T^d \neq 0$; otherwise both $\delta(T)$ and $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ are infinite. Let $M = R(T^d) = R(T^\infty)$. It is then easy to verify that T_M is surjective and $T_{X/M}$ is nilpotent. By Lemma 3, there is positive real number δ such that

$$\delta = \lim_{n\to\infty} \gamma(T^n)^{1/n} = \lim_{n\to\infty} \gamma(T_M^n)^{1/n}.$$

We know from [8] that δ is the surjectivity radius of T_M. We proceed to show that $\delta = \delta(T)$. If $0 < |\lambda| < \delta$, then $(T - \lambda I)_M$ is surjective. Since $T_{X/M}$ is nilpotent, $(T - \lambda I)_{X/M}$ is invertible and hence bounded below. Using Lemma 5, we deduce that $T - \lambda I$ is s-regular. Hence, $\delta \leq \delta(T)$.

Conversely, assume $0 < |\lambda| < \delta(T)$. Since $T_{X/M}$ is nilpotent, $(T - \lambda I)_{X/M}$ is invertible and $(T - \lambda I)^{-1}M = M$. It follows that

$$(T - \lambda I)M = (T - \lambda I)(T - \lambda I)^{-1}M = M \cap R(T - \lambda I).$$

It is routine to verify that T is quasi-Fredholm. Thus, $M = R(T^\infty) \subseteq R(T - \lambda I)$ by Lemma 6. So we have $(T - \lambda I)M = M$. This shows that the surjectivity radius of T_M is no less than $\delta(T)$. Hence, $\delta(T) \leq \delta$. □

A dual to the above theorem is the following.

Theorem 9. *If* $\operatorname{asc}(T) = d$ *and* $R(T^n)$ *is closed for all* $n \geq d$*, then the stability radius of* T *is given by* $\delta(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}$.

Theorem 10. *Let* T *be a quasi-Fredholm operator; then the stability radius of* T *is given by* $\delta(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}$.

Proof. Let $M = R(T^\infty) + N(T^d)$, $N = R(T^\infty)$. It follows from Theorem 7 that the subspaces M and N satisfy the requirement for Lemma 4. We now have:

(a) $T_{X/M}$ is bounded below and $\delta(T_{X/M}) = \lim_{n\to\infty} \gamma(T^n_{X/M})^{1/n}$.

(b) By Lemma 4 (iii), T_M satisfies the requirement for Theorem 8 and

$$\delta(T_M) = \lim_{n\to\infty} \gamma(T^n_M)^{1/n} = \lim_{n\to\infty} \gamma(T^n_N)^{1/n}.$$

(c) By (a), (b) and Lemma 4 (v), the limit $\delta = \lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and

$$\delta = \min\{\, \delta(T_{X/M}), \delta(T_M)\,\}.$$

If $0 < |\lambda| < \delta$, then $|\lambda|$ is less than both $\delta(T_{X/M})$ and $\delta(T_M)$. Thus $(T - \lambda I)_{X/M}$ is bounded below [8] and $(T - \lambda I)_M$ is surjective (see the remark for Theorem 8). Lemma 5 shows that $T - \lambda I$ is s-regular. Hence, $\delta \leq \delta(T)$.

Conversely, let $0 < |\lambda| < \delta(T)$. By Theorem 6, $M = R[(T - \lambda I)^\infty]$. Since $T - \lambda I$ is s-regular, it follows from Theorem 5 that $(T - \lambda I)_{X/M}$ is bounded below and $(T - \lambda I)_M$ is surjective. Therefore,

$$|\lambda| < \min\{\, \delta(T_{X/M}), \delta(T_M)\,\} = \delta.$$

This shows that $\delta(T) \leq \delta$. □

So far, the results obtained are independent of the Apostol representation of operators given in [12]. For the rest of this paper, we assume the following results from [12]. Let

$$\mathcal{N} = \bigvee_{\lambda\in\rho_\gamma(T)} N(T - \lambda I) \quad \text{and} \quad \mathcal{R} = \bigcap_{\lambda\in\rho_\gamma(T)} R(T - \lambda I).$$

So $\mathcal{N}$ is the closed subspace generated by $N(T - \lambda I)$ and $\mathcal{R}$ is the intersection of $R(T - \lambda I)$ over all $\lambda \in \rho_\gamma(T)$. Then $\mathcal{N}, \mathcal{R}$ are hyper-invariant subspaces of T with $\mathcal{N} \subseteq \mathcal{R}$. If T_δ, T_0, T_π are the maps induced by T on the spaces $\mathcal{N}, \mathcal{R}/\mathcal{N}, X/\mathcal{R}$ respectively, then the following properties hold.

(i) T_δ has dense range, T_π is injective.

(ii) T is s-regular if and only if T_δ is surjective, T_0 is invertible and T_π is bounded below.

These two facts are required for the proofs of the following theorems. In particular, we need to know that $\sigma(T_0) \subseteq \sigma_\gamma(T)$, which is a consequence of (ii). It is also known that T is quasi-Fredholm if and only if T_δ is surjective, T_0 is chain finite and T_π is bounded below. However, we will not use this fact.

Lemma 11. *Let* $N \subseteq M$ *be closed* T*-invariant subspaces of* X*. Suppose that* $T_{X/M}$ *is injective,* $T^d M \subseteq N$ *for some* d *and* T_N *has dense range. If the limit* $\limsup_{n\to\infty} \gamma(T^n)^{1/n}$ *is non-zero, then* T *is quasi-Fredholm.*

Proof. In the light of Theorem 7, it suffices to prove that $T_{X/M}$ is bounded below and T_N is surjective. It is clear from Lemma 1 that both $\limsup_{n\to\infty} \gamma(T^n_{X/N})^{1/n}$ and $\limsup_{n\to\infty} \gamma(T^n_M)^{1/n}$ are positive.

Since $T^d M \subseteq N$ and $T_{X/M}$ is injective, we have the inclusions

$$M \subseteq T^{-n}N \subseteq T^{-n}M \subseteq M$$

for each $n \geq d$. Hence $M = T^{-n}N$ and $N(T^n_{X/N}) = M/N$ for $n \geq d$. Applying Lemma 2 to the operator $T_{X/N}$, we deduce that $(T_{X/N})^{\wedge}$ and hence $T_{X/M}$ is bounded below.

Since T_N has dense range, so has T^n_N for each $n \geq d$. Therefore

$$N \subseteq \overline{T^n N} \subseteq \overline{T^n M} \subseteq \overline{N} = N.$$

Hence, $\overline{R(T^n_M)} = \overline{R(T^d_M)} = N$ for $n \geq d$. Applying Lemma 3 to the operator T_M, we deduce that $(T_M)^{\sim} = T_N$ is surjective. □

The proofs of the following two lemmas are elementary, so we omit them.

Lemma 12. *Let $N \subseteq M$ be closed T-invariant subspaces of X. If both $T_{X/M}$ and $T_{M/N}$ are injective, then $T_{X/N}$ is also injective.*

Lemma 13. *Let M_1, M_2 be T-invariant closed subspaces of X such that $M_1 + M_2$ is closed. Let*

$$X_1 = \frac{M_1}{M_1 \cap M_2},\ X_2 = \frac{M_2}{M_1 \cap M_2}, \quad Y_1 = \frac{M_1 + M_2}{M_1},\ Y_2 = \frac{M_1 + M_2}{M_2}.$$

Then the diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & Y_2 \\ {\scriptstyle T_{X_1}}\downarrow & & \downarrow{\scriptstyle T_{Y_2}} \\ X_1 & \xrightarrow{\varphi} & Y_2 \end{array} \qquad \begin{array}{ccc} X_2 & \xrightarrow{\psi} & Y_1 \\ {\scriptstyle T_{X_2}}\downarrow & & \downarrow{\scriptstyle T_{Y_1}} \\ X_2 & \xrightarrow{\psi} & Y_1 \end{array}$$

are commutative, and φ and ψ are linear homeomorphisms induced by the identity on X.

We now give a generalization of [1, Prop. 3.3].

Theorem 14. *Let σ be a closed and open subset of $\sigma_\gamma(T)$. If $0 \in \sigma$ and*

$$\limsup_{n\to\infty} \gamma(T^n)^{1/n} = r > \sup_{\lambda\in\sigma}|\lambda|, \tag{6}$$

then T is quasi-Fredholm.

Proof. Since T_δ has dense range and T_π is injective, we can apply both part (i) and (ii) of Lemma 1 to show that $\gamma(T^n_0) \geq \gamma(T^n)$ for all n. Let $\sigma_1 = \sigma \cap \sigma(T_0)$, $\sigma_2 = [\sigma_\gamma(T) \setminus \sigma] \cap \sigma(T_0)$; then both σ_1 and σ_2 are closed. Since $\sigma(T_0) \subseteq \sigma_\gamma(T)$, we have $\sigma_1 \cup \sigma_2 = \sigma(T_0)$. Hence σ_1 and σ_2 are spectral sets of T_0. The spectral sets induce a decomposition

$$\mathcal{R}/\mathcal{N} = X_1 \oplus X_2, \quad T_0 = T_1 \oplus T_2 \quad \text{with } \sigma(T_1) = \sigma_1,\ \sigma(T_2) = \sigma_2.$$

Since $0 \notin \sigma_2$, T_2 is invertible. We claim that T_1 is nilpotent. Let us assume that $\sigma_1 \neq \emptyset$; otherwise, $X_1 = 0$ and T_1 is trivially nilpotent. For each $x_1 \in X_1$, $x_2 \in X_2$,

$$T_0(x_1 + x_2) \in X_1 \Rightarrow T_2 x_2 \in X_1 \Rightarrow T_2 x_2 \in X_1 \cap X_2 = \{0\} \Rightarrow x_2 = 0.$$

Therefore, $T_0^{-1} X_1 \subseteq X_1$. Thus T_0 and X_1 satisfy part (i) of Lemma 1. Hence, $\gamma(T^n_1) \geq \gamma(T^n_0) \geq \gamma(T^n)$ for all n. Taking limits,

$$r = \limsup_{n\to\infty} \gamma(T^n)^{1/n} \leq \limsup_{n\to\infty} \gamma(T^n_1)^{1/n}.$$

It is well-known that if $S \neq 0$, then $\gamma(S) \leq \|S\|$. So if T_1 is not nilpotent, then

$$r \leq \limsup_{n\to\infty} \gamma(T_1^n)^{1/n} \leq \lim_{n\to\infty} \|T_1^n\|^{1/n} = \sup\{|\lambda| : \lambda \in \sigma_1\}.$$

Note that the equality on the right comes from the spectral radius formula and the fact that $\sigma(T_1) = \sigma_1$. This contradicts (6). Hence, T_1 is nilpotent. This proves our claim.

Suppose $X_1 = M_1/\mathcal{N}$ and $X_2 = M_2/\mathcal{N}$. Since $\mathcal{R}/\mathcal{N} = X_1 \oplus X_2$, it is not difficult to see that

$$M_1 \cap M_2 = \mathcal{N} \quad \text{and} \quad M_1 + M_2 = \mathcal{R}.$$

In the context of Lemma 13, we have $T_{Y_1} = T_{\mathcal{R}/M_1}$, $T_{X_2} = T_2$ and $T_{\mathcal{R}/M_1} = \psi T_2 \psi^{-1}$. Since T_2 is invertible, so is $T_{\mathcal{R}/M_1}$. By definition, $T_{X/\mathcal{R}}$ equals T_π, which is always injective. Thus, both $T_{X/\mathcal{R}}$ and $T_{\mathcal{R}/M_1}$ are injective, and so is T_{X/M_1} by Lemma 12. Moreover, $T_{M_1/\mathcal{N}} = T_1$ is nilpotent and $T_{\mathcal{N}} = T_\delta$ have dense range. Using Lemma 11, we deduce that T is quasi-Fredholm. □

Corollary 15. *Let* 0 *be an isolated point of* $\sigma_\gamma(T)$*; then the limit* $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ *always exists. This limit is non-zero if and only if* T *is quasi-Fredholm.*

Proof. Let $\eta = \limsup_{n\to\infty} \gamma(T^n)^{1/n}$. We have two cases.

(a) If $\eta = 0$, then $\lim_{n\to\infty} \gamma(T^n)^{1/n} = 0$.

(b) If $\eta > 0$, we can put $\sigma = \{0\}$, and T is quasi-Fredholm by Theorem 14. Hence, the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists, and is equal to the stability radius of T by Theorem 10.

Therefore, the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ always exists and equals η. If $\eta \neq 0$, then case (b) must be true and T is quasi-Fredholm. Conversely, if T is quasi-Fredholm, then η is the stability radius of T. From Lemma 6, we know that the stability radius of a quasi-Fredholm operator is always non-zero. This completes our proof. □

The significance of the corollary is that the class of quasi-Fredholm operators is the most general class of operators for which the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ is equal to the stability radius $\delta(T)$ with respect to the Apostol spectrum.

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