PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 1231–1236
S 0002-9939(98)04278-6

AN APPLICATION OF THE LEFSCHETZ FIXED-POINT THEOREM TO NON-CONVEX DIFFERENTIAL INCLUSIONS ON MANIFOLDS

STANISŁAW DOMACHOWSKI AND TADEUSZ PRUSZKO

(Communicated by Hal L. Smith)

Abstract. A selector theorem for non-convex orientor fields on closed manifolds is given and the Lefschetz fixed point theorem is used to establish an existence result for these ones.

1. Introduction

In this note we give an existence theorem for the initial value problem associated with the following differential inclusion

$$\dot{x}(t) \in F(t, x) \tag{1}$$

where $F\colon [a, b] \times M^n \to TM^n$ is a multivalued map such that for every real number t from an interval $[a, b]$ and for every point p from a manifold M^n (modeled on Euclidean space $\mathbf{R}^n$) $F(t, p)$ is a non-empty and closed subset in the fibre T_pM^n. The existence problem for differential inclusions in Euclidean space ($M^n = \mathbf{R}^n$) with convex-valued right-hand side has a long history (see [9] for details). In the case when $F\colon [a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is a multivalued map, not necessarily a convex-valued one, some existence theorems have been given by Filippov [4] for F continuous with respect to Hausdorff distance and Kaczyński and Olech [7] (see also Olech [9]) for F satisfying hypotheses of Carathéodory type. The solutions have been obtained by using some approximate techniques there.

A useful tool for differential inclusions turned out to be the selector theorems given by Antosiewicz and Cellina [1], Łojasiewicz (Jr) [8] and far-reaching of their generalizations due to Fryszkowski [5] and Bressan and Colombo [2]. These theorems enable us to apply the Schauder fixed point theorem to establish some existence results for non-convex differential inclusions with right-hand side satisfying hypotheses of Carathéodory type in Euclidean space [1] or with a lower semicontinuous one [8]. Moreover the selector theorems enable us to apply the topological degree methods to differential boundary value problems for non-convex differential inclusions [10]. In this note we prove a selector theorem for non-convex orientor fields from the product $[a, b] \times M^n$ into the tangent bundle TM^n that are lower

Received by the editors September 23, 1996.

1991 *Mathematics Subject Classification.* Primary 58F32; Secondary 47H04, 28B20.

The work of the first author was supported by the University of Gdańsk, grant nr BW-5100-5-0055-6.

semicontinuous or satisfy Carathéodory conditions. Then using the Lefschetz fixed point theorem for ANR's (cf. [6]) we obtain as a consequence a solution of (1).

2. Definitions and theorem

Let $\mathcal{A} = \{(U_1, \varphi_1), \dots, (U_k, \varphi_k)\}$ be an atlas on a closed C^2-manifold M^n and let $\tilde{\mathcal{A}} = \{(\tilde{U}_1, \tilde{\varphi}_1), ..., (\tilde{U}_k, \tilde{\varphi}_k)\}$ be an atlas on the tangent bundle TM^n with diffeomorphisms $\tilde{\varphi}_i\colon \tilde{U}_i \to \varphi_i(U_i) \times \mathbf{R}^n$ $(i = 1, \dots, k)$ given by $\tilde{\varphi}_i([l]) = (\varphi(p), (\lambda_i^p)^{-1}([l]))$ for $[l] \in TM^n, p \in U_i$ where $\lambda_i^p\colon \mathbf{R}^n \to T_pM^n$ is a linear isomorphism such that for every C^1-path $l\colon (-\varepsilon, \varepsilon) \to M^n$ with $l(0) = p$ we have $\lambda_i^p[(\varphi_i \circ l)'(0)] = [l]$ ($(\varphi_i \circ l)'(0)$ is the gradient of $(\varphi_i \circ l)\colon (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ at the point $0 \in (-\varepsilon, \varepsilon)$). For a map $x\colon [a, b] \to M^n$ we accept

$$\dot{x}(t) = \lambda_i^{x(t)}[(\varphi \circ x)'(t)] \qquad \text{in} \quad t \in (a, b)$$

provided $x(t) \in U_i$ and $(\varphi_i \circ x)\colon [a, b] \to \mathbf{R}^n$ is a differentiable map at the point t. A multivalued map $F\colon [a, b] \times M^n \to TM^n$ we call an orientor field if for every $(t, p) \in [a, b] \times M^n$ the set $F(t, p) \subset T_pM^n$ is nonempty and closed in T_pM^n. The map $F\colon [a, b] \times M^n \to TM^n$ is lower semicontinuous (l.s.c.) if for every closed set $X \subset TM^n$ the counterimage $F^{-1}(X) = \{(t, p) : F(t, p) \subset X\}$ is closed in $[a, b] \times M^n$. We say that $F\colon [a, b] \times M^n \to TM^n$ satisfies Carathéodory conditions if for each $t \in [a, b]$ the map $F(t, \cdot)\colon M^n \to TM^n$ is continuous and for each $p \in M^n$ the map $F(\cdot, p)\colon [a, b] \to TM^n$ is measurable (i.e. the set $\{t \in [a, b] : F(t, p) \cap U \neq \emptyset\}$ is measurable for any open set $U \subset TM^n$). An orientor field $F\colon [a, b] \times M^n \to TM^n$ is integrable bounded if there exists an integrable function $\eta\colon [a, b] \to \mathbf{R}$ such that $|(\lambda_i^p)^{-1}(F(t, p))| \leq \eta(t)$ for each $(t, p) \in [a, b] \times U_i$ and $i = 1, \dots, k$ ($|Y|$ denotes the Hausdorff distance of the sets $Y \subset \mathbf{R}^n$ and $\{0\} \subset \mathbf{R}^n$). For an orientor field $F\colon [a, b] \times M^n \to TM^n$ we formulate an initial value problem

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. on } [a, b], \\ x(a) = p. \end{cases} \tag{1}$$

A solution of (1) we call a continuous function $x\colon [a, b] \to M^n$ such that the correspondence $[a, b] \ni t \to \dot{x}(t) \in TM^n$ is defined a.e. on $[a, b]$ and it is a measurable map.

Theorem 1. *Suppose an integrable bounded orientor field $F\colon [a, b] \times M^n \to TM^n$ satisfies the Carathéodory conditions or it is an l.s.c. map. Then there exists at least one solution of problem* (1).

3. Integrable selectors

Let $C([a, b], M^n)$ be a space of all continuous functions (paths) $x\colon [a, b] \to M^n$ with the supremum metric, let $\mathcal{M}([a, b], TM^n)$ be a space of all measurable single-valued functions $y\colon [a, b] \to TM^n$ and let $\{W_i\}_{i=1}^k$ be an open cover of M^n such that $W_i \subset \overline{W_i} \subset U_i$ $(i = 1, \dots, k)$. We define operators $\Lambda_i\colon \mathcal{M}([a, b], TM^n) \to \mathcal{M}([a, b], \mathbf{R}^n)$ given by

$$\Lambda_i(y)(t) = \begin{cases} (\lambda_i^{\pi(y(t))})^{-1}(y(t))\beta_i(\pi(y(t))) & \text{if } y(t) \in \tilde{W}_i, \\ 0 & \text{if } y(t) \notin \tilde{W}_i, \end{cases} \qquad (i = 1, \dots, k) \tag{3.1}$$

where $\{\beta_i\}_{i=1}^k$ is a partition of unity on M^n such that $\operatorname{supp}\beta_i \subset W_i$ and $\pi\colon TM^n \to M^n$ is the canonical projection such that $\pi[l] = p$ for $[l] \in T_pM^n$ and $\tilde{W}_i = \tilde{\varphi}_i^{-1}(\varphi(W_i) \times \mathbf{R}^n)$.

In what follows for an orientor field $F\colon [a,b] \times M^n \to TM^n$ and for a path $x\colon [a,b] \to TM^n$ we say a function $y \in \mathcal{M}([a,b], TM^n)$ is an integrable selector of $F \circ (id \otimes x)\colon [a,b] \to TM^n$ iff $y(t) \in F(t, x(t))$ a.e. on $[a,b]$ and $\Lambda_i(y)$ is integrable for each $i = 1, \dots, k$. The integrable selector y of $F \circ (id \otimes x)$ we will denote by $y \in F \circ (id \otimes x)$.

Lemma 1. *Assume $F\colon [a,b] \times M^n \to TM^n$ is an integrable bounded orientor field that satisfies the Carathéodory conditions or it is l.s.c.. Then there exists a single-valued map $f\colon C([a,b], M^n) \to \mathcal{M}([a,b], TM^n)$ such that $f(x) \in F \circ (id \otimes x)$ for every $x \in C([a,b], M^n)$ and the compositions $(\Lambda_i \circ f)\colon C([a,b], M^n) \to L^1([a,b], \mathbf{R}^n)$ are continuous for each $i = 1, \dots, k$.*

Proof. Let us notice that without loss of generality we can assume that the atlas $\mathcal{A} = \{(U_i, \varphi_i) : \ i = 1, \dots, k\}$ on M^n satisfies conditions $\varphi_i(U_i) \subset B(i, \frac{1}{4})$ for $i = 1, \dots, k$ with $B(i, \frac{1}{4}) = \{v \in \mathbf{R}^n : |v - i| < \frac{1}{4}\}$. Let $x\colon [a,b] \to M^n$ be a continuous map. We say a system $D^x = (D_1^x, \dots, D_k^x)$ of measurable subsets of $[a,b]$ is a decomposition of the interval $[a,b]$ with respect to x iff the Lebesgue measure $\mu(\bigcup_{i=1}^k D_i^x) = b - a$, $D_i^x \cap D_j^x = \emptyset$ for $i \neq j$ and $x(D_i^x) \subset W_i$ for each $i = 1, \dots, k$. With every $y \in \mathcal{M}([a,b], TM^n)$ we will associate a measurable map $z(y, D^x, \cdot)\colon [a,b] \to \mathbf{R}^n \times \mathbf{R}^n$ given by $z(y, D^x, t) = \tilde{\varphi}_i(y(t))$ for $t \in D_i^x$ where $D^x = (D_1^x, \dots, D_k^x)$ is a decomposition of $[a,b]$ with respect to the function $x \in C([a,b], M^n)$. Now we can give a sketch of the proof.

1) The multivalued map $\Phi\colon C([a,b], M^n) \to L^1([a,b], \mathbf{R}^n \times \mathbf{R}^n)$ given by

$$\Phi(x) = \{z(y, D^x, \cdot) : \quad y \in F \circ (id \otimes x) \quad \text{and} \quad D^x \text{ is a decomposition of } [a,b]\}$$

is l.s.c. with decomposable values $\Phi(x)$ for each $x \in C([a,b], M^n)$.

2) By virtue of the selection theorem, due to Bressan-Colombo [2], there exists a continuous single-valued map $g\colon C([a,b], M^n) \to L^1([a,b], \mathbf{R}^n \times \mathbf{R}^n)$ such that $g(x) \in \Phi(x)$ for each $x \in C([a,b], M^n)$.

3) Finally the single-valued map $f\colon C([a,b], M^n) \to \mathcal{M}([a,b], TM^n)$ given by $f(x)(t) = (\tilde{\varphi}_i^{-1})(g(x)(t))$ for $t \in D_i^x$ $(i = 1, \dots, k)$, is the desired one.

First of all we notice that $\Phi(x) \neq \emptyset$. Indeed for a sequence of points $a = t_0 < t_1 < \dots < t_m = b$ together with a function $\tau\colon \{1, \dots, m\} \to \{1, \dots, k\}$ for that $x([t_{j-1}, t_j] \subset W_{\tau(j)}$ there are measurable maps $y_j\colon [t_{j-1}, t_j] \to \tilde{U}_{\tau(j)} \subset TM^n$ such that $y_j(t) \in F(t, x(t))$ a.e. on $[t_{j-1}, t_j]$ $(j = 1, \dots, m)$. Let $y\colon [a,b] \to TM^n$ be a measurable function defined by $y_{|[t_{j-1}, t_j]} = y_j$ a.e. on $[t_{j-1}, t_j]$ for $j = 1, \dots, m$ and let $D^x = (D_1^x, \dots, D_k^x)$ be a decomposition of $[a,b]$ given by $D_i^x = \bigcup_{\tau(j)=i} [t_{j-1}, t_j)$ for $i \neq \tau(m)$ and $D_{\tau(m)}^x = \bigcup_{\tau(j)=\tau(m)} [t_{j-1}, t_j) \cup \{t_m\}$. Then we have $z(y, D^x, \cdot) \in \Phi(x)$. Next we show the lower semicontinuity of Φ. Let x be a limit in $C([a,b], M^n)$ of a sequence $\{x_l\} \subset \Phi^{-1}(K)$ where K is a closed subset in $L^1([a,b], \mathbf{R}^n \times \mathbf{R}^n)$ and let $v \in \Phi(x)$. We have to show that $v \in K$. By definition of Φ we see that $v = z(y, D^x, \cdot)$ with a decomposition $D^x = (D_1^x, \dots, D_k^x)$ and $y \in F \circ (id \otimes x)$ such that $v(t) = \tilde{\varphi}_i(y(t))$ for $t \in D_i^x$ $(i = 1, \dots, k)$. For every integer $p \in \mathbf{N}$ there exist subsets $B_i^p \subset D_i^x$ $(i = 1, \dots, k)$ that are closed in $[a,b]$ and $\mu(D_i^x \backslash B_i^p) < \frac{1}{p}$. Without loss of generality we can assume that $x_p(B_i^p) \subset W_i$ for each $p \in \mathbf{N}$ and $i = 1, \dots, k$. Let $z(y_p, E^{x_p}, \cdot) \in \Phi(x_p)$ with a decomposition $E^{x_p} = (E_1^{x_p}, \dots, E_k^{x_p})$, let $D^{x_p} = (D_1^{x_p}, \dots, D_k^{x_p})$ be another decomposition with respect to x_p given by $D_i^{x_p} = B_i^p \cup (C^p \cap E_i^{x_p})$ $(i = 1, \dots, k)$ where $C^p = [a,b] \backslash \bigcup_{i=1}^k B_i^p$, and let $d_p(t)$ be the distance of the point $v(t)$ to the set $\tilde{\varphi}_i(F(t, x_p(t)))$ for $t \in D_i^{x_p}, i = 1, \dots, k$ and $p \in \mathbf{N}$.

For every $p \in \mathbf{N}$ the map $F_p \colon [a,b] \to \mathbf{R}^n \times \mathbf{R}^n$ given by

$$F_p(t) = \begin{cases} B(v(t), 2d_p(t)) \cap \tilde{\varphi}_i(F(t, x_p(t))) & \text{if} \quad d_p(t) > 0 \quad \text{and} \quad t \in D_i^{x_p}, \\ v(t) & \text{if} \quad d_p(t) = 0 \quad \text{or} \quad t \notin \bigcup_{i=1}^k D_i^{x_p}, \end{cases}$$

is measurable and so there exists a sequence $\{v_p\}$ of measurable functions $v_p \colon [a,b] \to \mathbf{R}^n \times \mathbf{R}^n$ such that $v_p(t) \in \tilde{\varphi}_i(F(t, x_p(t)))$ a.e. on $D_i^{x_p}$ $(i = 1, \dots, k)$ and $v_p \in \Phi(x_p) \subset K$. Moreover $\|v_p - v\| \to 0$ because $F \colon [a,b] \times M^n \to TM^n$ is integrable bounded and satisfies Carathéodory conditions or it is l.s.c.. Therefore $v \in \Phi$.

Now let us see that for any two functions $z_1(y_1, D^x, \cdot), z_2(y_2, C^x, \cdot) \in \Phi(x)$ we have $z_1 \cdot \chi_A + z_2 \cdot \chi_{A'} \in \Phi(x)$ where χ_A is the characteristic function of a measurable set $A \subset [a,b]$ and $A' = [a,b] \backslash A$. Indeed for the decompositions $D^x = (D_1^x, \dots, D_k^x), C^x = (C_1^x, \dots, C_k^x)$ and $B^x = (B_1^x, \dots, B_k^x)$ with $B_i^x = (D_i^x \cap A) \cup (C_i^x \cap A')$ and for $y \in F \circ (id \otimes x)$ given by

$$y(t) = \begin{cases} y_1(t) & \text{for } t \in A, \\ y_2(t) & \text{for } t \in A', \end{cases}$$

we obtain $z_1 \cdot \chi_A + z_2 \cdot \chi_{A'} = z(y, B^x, \cdot) \in \Phi(x)$ where χ_A is the characteristic function of A. Therefore for each $x \in C([a,b], M^n)$ the set $\Phi(x)$ is decomposable. Finally we prove that for a single-valued continuous map $g \colon C([a,b], M^n) \to L^1([a,b], \mathbf{R}^n \times \mathbf{R}^n)$ given in (2) the single-valued maps $(\Lambda_i \circ f) \colon C([a,b], M^n) \to L^1([a,b], \mathbf{R}^n)$, where f is given in (3) for the map g are continuous for each $i = 1, \dots, k$. For this let $\{x_n\}$ be a sequence convergent to x in $C([a,b], M^n)$ and let $Q = x^{-1}(U_i)$. Then

$$\begin{aligned} &\int_{[a,b]} |\Lambda_i \circ f(x_m)(t) - \Lambda_i \circ f(x)(t)| \, dt \\ &\qquad \leq \sum_{j=1}^k \int_{Q \cap D_j^{x_m} \cap D_j^x} |(\lambda_i^{x_m(t)})^{-1} \tilde{\varphi}_j^{-1}(g(x_m)(t)) \beta_i(x_m(t)) \\ &\qquad\qquad - (\lambda_i^{x(t)})^{-1} \tilde{\varphi}_j^{-1}(g(x)(t)) \beta_i(x(t))| dt \\ &\qquad + \sum_{j \neq s} \int_{Q \cap D_j^{x_m} \cap D_s^x} 2\eta(t) dt. \end{aligned}$$

Since by the continuity of g

$$\sum_{s,j=1}^k \int_{D_j^{x_m} \cap D_s^x} |\varphi_j(x_m(t)) - \varphi_s(x(t))| dt \to 0;$$

hence the measures $\mu(D_j^{x_m} \cap D_s^x) \to 0$ for $s \neq j$ because $|\varphi_j(x(t)) - \varphi_s(x(t))| > \frac{1}{4}$ for $s \neq j$. Now again the continuity of g together with the above inequality implies

$$\int_{[a,b]} |\Lambda_i \circ f(x_m)(t) - \Lambda_i \circ f(x)(t)| \, dt \to 0 \quad \text{for} \quad i = 1, \dots, k.$$

The proof is completed. □

4. The proof of Theorem 1

Let $\eta \colon [a,b] \to \mathbf{R}$ be an integrable function such that $|(\lambda_i^p)^{-1}(F(t,p))| \leq \eta(t)$ for every $(t,p) \in [a,b] \times U_i$, $i = 1, \dots, k$, and let $j \colon M^n \to \mathbf{R}^{2n+1}$ be the Whitney imbedding. With every point $q \in M^n$ and positive real number $K > 0$ we associate a subspace $C_{q,K} \subset C([a,b], M^n)$ of all functions $x \colon [a,b] \to M^n$ such that $x(a) = p$ and $|j(x(t)) - j(x(\bar{t}))| \leq K \int_t^{\bar{t}} \eta(s) ds$ for every $t, \bar{t} \in [a,b]$. On the other hand let $f \colon C([a,b], M^n) \to \mathcal{M}([a,b], TM^n)$ be an integrable selector with respect to the cover $\{W_i\}_{i=1}^k$ given by Lemma 1 for the orientor field F and $\{\beta_i\}_{i=1}^k$ be a partition of the unity on M^n the same as in (3.1). We define a single-valued map $\psi \colon C_{q,K} \to C([a,b], M^n)$ such that for every function $x \in C_{q,K}$ the value $\psi(x)$ is the unique solution of the following initial value problem:

$$(2)_x \qquad \begin{cases} \dot{u}(t) = f_x(t, u(t)), \\ u(a) = q, \end{cases}$$

where $f_x(t,p) = \sum\limits_{i=1}^k f_i^x(t,p)$,

$$f_i^x(t,p) = \begin{cases} \alpha_i(p) \cdot \alpha_i(x(t)) \cdot \lambda_i^p \circ (\lambda_i^{x(t)})^{-1}[f(x)(t)] & \text{if } \ p \in W_i \ \text{ and } \ f(x)(t) \in \tilde{U}_i, \\ 0 & \text{if } \ p \notin W_i \ \text{ or } \ f(x)(t) \notin \tilde{U}_i \ , \end{cases}$$

and

$$\alpha_i = \beta_i \cdot \left[\sum_{i=1}^k \beta_i^2 \right]^{\frac{-1}{2}} .$$

By virtue of (3.1), (5.2) and (5.4) respectively in [3] we obtain the following:

(i) the definition of ψ is correct;

(ii) the space $C_{q,K}$ is a compact *ANR*;

(iii) there exists a real number $K > 0$ such that $\psi(C_{q,K}) \subset C_{q,K}$.

Moreover by using both Lemma 1 and the continuous dependence of solutions of $(2)_x$ on the vector fields f_x the map ψ is continuous. Now by the Lefschetz fixed-point theorem there exists a function $x \in C_{q,K}$ such that $x = \psi(x)$.

Finally we can see that the fixed point x of ψ satisfies

$$\begin{cases} \dot{x}(t) = f(x)(t) \quad \text{a.e. on } [a,b], \\ x(a) = q, \end{cases}$$

and so we have a solution of (1). The proof is completed.

References

[1] H.A. Antosiewicz and A. Cellina, Continuous selections and differential relations, J. Differential Equations, 19 (1975), pp. 386 - 398. MR **55:**3373

[2] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, Studia Math., 90 (1988), pp. 70 - 85. MR **89j:**54021

[3] S. Domachowski and T. Pruszko, An application of the Eilenberg - Montgomery theorem to measurable orientor fields on manifolds, Funkcialaj Ekvacioj, 36 (1993), pp. 95 - 107. MR **94e:**58008

[4] A.F. Filippov, The existence of solutions of multivalued differential equations (in Russian), Mat. Zametki, 10 (1971), pp. 307 - 313. MR **45:**7142

[5] A. Fryszkowski, Continuous selections for a class of non-convex multivalued maps, Studia Math., 76 (1983), 163 - 174. MR **85j:**54022

[6] A. Granas, Points fixes pour les applications compactes: espaces de Lefschetz et théorie de l'indice, Notes de cours, Séminaire de Mathematiques Supérieurs, Montréal, 1973. [Les Press de l'Université de Montréal, 1980.] MR **81i:**55002

[7] H. Kaczyński and C. Olech, Existence of solutions of orientor fields with non-convex right-hand side, Ann. Polon. Math. 29 (1974), pp. 61 - 66. MR **50:**10974

[8] S. Łojasiewicz (Jr), The existence of solutions for lower semicontinuous orientor fields, Bull. Acad. Polon. Sci., **28**, no. 9-10 (1980), pp. 483-487. MR **82j:**34013

[9] C. Olech, Existence of solutions of non-convex orientor fields, Bull Unione Math. Ital., 11 (1975), pp. 189 - 197. MR **52:**15174

[10] T. Pruszko, Some applications of the topological degree theory to multivalued boundary value problems, Dissertationes Math. 229 (1984). MR **85j:**34030

Institute of Mathematics, Gdańsk University, ul. Wita Stwosza 57, 80–952 Gdańsk, Poland