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TORI IN CERTAIN ASPHERICAL FOUR–MANIFOLDS

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ABSTRACT. The homology classes represented by embedded or mapped tori in the product of two surfaces are completely characterized.

If X^4 is a 4-manifold, then any element $\xi \in H_2(X^4)$ can be represented by a map from a closed, oriented surface, and also by an embedded, closed, oriented surface (perhaps of bigger genus). If X^4 is simply connected then $H_2(X^4) \approx \pi_2(X^4)$ by the Hurewicz Theorem, so that every class ξ is represented by a map $S^2 \to X^4$. It has been an intensively studied fundamental problem to determine which classes are represented by smooth or topologically embedded 2-spheres, and, more generally, to determine the minimum genus of an embedded surface representing ξ. In general there are striking differences between the smooth and topological categories. See, for example, [2] and [3] for some of the latest results in this area.

In this note we begin to address similar questions in the aspherical case, in which there are no interesting 2-spheres at all, apart from "local knot theory."

We determine which 2-dimensional homology classes can be represented by embedded tori in $X^4 = F^2 \times G^2$, where F^2 and G^2 are closed orientable surfaces of genus greater than 1. It turns out that there is no difference between the topological and differentiable categories in this case.

Theorem 1. *Let F^2 and G^2 be closed orientable surfaces of genus greater than 1. An element $\xi \in H_2(F^2 \times G^2)$ is represented by a topological or differentiable embedded torus $T^2 \subset F^2 \times G^2$ if and only if $\xi = \alpha \times \beta$ for some $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$.*

This follows immediately from the following two more precise propositions.

Proposition 1. *Any map $f : T^2 \to F^2 \times G^2$ represents a homology class of the form $\alpha \times \beta$ for some $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$.*

Proposition 2. *If $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$ are given, then the homology class $\alpha \times \beta$ is represented by a smooth embedding $f : T^2 \to F^2 \times G^2$.*

Lemma 1. *If F^2 is a surface of negative euler characteristic, then any two commuting elements in $\pi_1(F^2)$ generate a cyclic subgroup of $\pi_1(F^2)$.*

Proof. Let $\alpha, \beta \in \pi_1(F^2)$ be commuting elements and let $S \to F^2$ be the covering map, where $\pi_1(S)$ corresponds to the subgroup of $\pi_1(F^2)$ generated by α and β. If it is a finite-sheeted covering, then $S \cong T^2$ by the classification of surfaces, and

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hence $\chi(S) = 0$. But multiplicativity of the euler characteristic then implies that $\chi(F^2) = 0$ too, a contradiction. Therefore the covering must be an infinite-sheeted covering. In particular, S is non-compact, and hence has a free fundamental group. Since $\pi_1(S)$ is abelian, this implies that it is cyclic, as required. □

Corollary 1. *If F^2 is a closed orientable surface of negative euler characteristic, and $f : T^2 \to F^2$ is a map, then there is a primitive homology class $\alpha \in H_1(F^2)$ such that the image of $f_* : H_1(T^2) \to H_1(F^2)$ is contained in the cyclic subgroup generated by α.*

Proof. Since $\pi_1(T^2) \approx H_1(T^2)$, the image of $H_1(T^2)$ in $H_1(F^2)$ is the image under the Hurewicz homorphism of the image of $\pi_1(T^2)$ in $\pi_1(F^2)$, which is cyclic by Lemma 1, and hence contained in a cyclic subgroup generated by a primitive element. □

We will need the following well-known result. For a proof, see [4].

Lemma 2. *If $\alpha \in H_1(F^2)$ is a nonzero homology class, then α is represented by a simple closed curve embedded in F^2 if and only if α is primitive.* □

Proof of Proposition 1. Let $f : T^2 \to F^2 \times G^2$ represent a nonzero homology class and let $p_F : F^2 \times G^2 \to F^2$ and $p_G : F^2 \times G^2 \to G^2$ denote the projections onto the two factors. By Corollary 1 the image of $(p_F \circ f)_* : H_1(T^2) \to H_1(F^2)$ is contained in a cyclic subgroup $\langle\alpha\rangle$ generated by a primitive element $\alpha \in H_1(F^2)$. Similarly, the image of $(p_G \circ f)_* : H_1(T^2) \to H_1(G^2)$ is contained in a cyclic subgroup $\langle\beta\rangle$ generated by a primitive element $\beta \in H_1(G^2)$. Now by Lemma 2 both α and β are represented by simple closed curves A and B in F^2 and G^2, respectively. It follows that $f_*[T^2]$ is a multiple of $[A \times B]$ in $H_2(F^2 \times G^2)$. But $n(\alpha \times \beta) = (n\alpha) \times \beta$, which is what is needed. □

Proof of Proposition 2. If we write $\alpha = m\alpha^*$ and $\beta = n\beta^*$, where α^* and β^* are primitive homology classes, then $\alpha \times \beta = mn\,\alpha^* \times \beta^* = (mn\,\alpha^*) \times \beta^*$. Now, by Lemma 2, α^* and β^* can be represented by simple closed curves A^* and B^*, respectively. Since A^* and B^* have trivial normal bundles, so does the torus $A^* \times B^* \subset F^2 \times G^2$. The proof then is completed by Lemma 3 below. □

Lemma 3. *The map $T^2 \to T^2 \times D^2$ given by $(z, w) \mapsto (z^k, w, 0)$ is homotopic to an embedding.*

Proof. Just define an explicit embedding by $(z, w) \mapsto (z^k, w, z)$ □

Remark 1. Lemma 3 generalizes to state that if F^2 is a closed oriented surface of genus g and n is a positive integer, then the homology class in $H_2(F^2 \times D^2)$ corresponding to $n[F^2 \times \{0\}]$ is represented by an embedding of the surface of genus $ng - (n - 1)$. This is the least possible genus that can even represent $n[F^2 \times \{0\}]$ by a map, by Kneser's inequality. For a modern exposition of Kneser's theorem, see [5].

Remark 2. Any map of nonzero degree $T^2 \to T^2$ is homotopic to a covering, and *any* covering $T^2 \to T^2 \times \{0\} \subset T^2 \times D^2$ is homotopic to an embedding. The proof is somewhat more difficult than that of Lemma 3 and depends on lifting a pair of commuting permutations σ, $\tau \in S_n$, that generate a transitive subgroup of S_n, to a pair of *commuting* braids b_σ and b_τ over them in the braid group B_n.

Remark 3. No nontrivial connected covering F^2 of the 0-section of a *nontrivial* oriented 2-disk bundle E over a closed orientable surface G^2 is homotopic to an embedding of F^2 in E. See [1], for example.

It follows from the results of this note that a map $T^2 \to F^2 \times G^2$ is homologous to an embedding $T^2 \to F^2 \times G^2$. Is it possible that a map $T^2 \to F^2 \times G^2$ is actually *homotopic* to an embedding? It seems unlikely.

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