PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 1043–1047
S 0002-9939(98)04296-8

REMARKS ON THE RESULTS BY KOSKELA CONCERNING THE RADIAL UNIQUENESS FOR SOBOLEV FUNCTIONS

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(Communicated by Albert Baernstein II)

ABSTRACT. In this note we aim to complete the results by Koskela concerning the radial uniqueness for Sobolev functions.

Let φ be a positive nonincreasing function on the interval $(0,\infty)$, and let $\mathbf{B}$ denote the unit ball of R^n. Consider a p-precise function u on $\mathbf{B}$ such that

$$\int_{U(\varepsilon)} |\nabla u(x)|^p dx \le \varepsilon^p \varphi(\varepsilon) \qquad \text{for any } \varepsilon > 0,$$

where $U(\varepsilon) = \{x \in \mathbf{B} : |u(x)| < \varepsilon\}$. We give conditions on φ which assure that $u = 0$ whenever u has vanishing fine boundary limits on a set of positive p-capacity.

We are also concerned with the sharpness.

1. Statement of results

For $1 < p < \infty$ and an open set $G \subset R^n$, we denote by $W^{1,p}(G)$ the Sobolev space of all functions u on G such that

$$\int_G |\nabla u(x)|^p dx < \infty,$$

where ∇ denotes the gradient. For a set $E \subset G$, we define

$$C_p(E;G) = \inf \int_G |\nabla u(x)|^p dx,$$

where the infimum is taken over all $u \in W^{1,p}(R^n)$ such that $u = 0$ outside G and

$$\int |x-y|^{1-n} |\nabla u(y)| dy \geqq 1 \qquad \text{for every } x \in E$$

(see [2], [7] and [9]). We write $C_p(E) = 0$ if $C_p(E \cap G; G) = 0$ for every bounded open set G.

Let $\mathbf{B}$ denote the unit ball in R^n. For each $u \in W^{1,p}(\mathbf{B})$, we can find an extension $u^* \in W^{1,p}(R^n)$ with compact support such that

$$u = u^* \qquad \text{for almost every } x \in \mathbf{B}$$

Received by the editors September 18, 1996.
1991 *Mathematics Subject Classification.* Primary 31B25, 31B15, 46E35.
Key words and phrases. p-precise functions, Sobolev functions, capacity, fine boundary limits.

and u^* is p-precise in the sense of Ziemer [10]. In view of [4], [5], we note further that u^* can be written as

$$u^*(x) = c \int \frac{x-y}{|x-y|^n} \cdot \nabla u^*(y)dy$$

with a constant c, where " $\cdot$ " denotes the usual inner product and the integrals are absolutely convergent p-q.e. (quasi everywhere) on R^n, that is, for every $x \in R^n - E$ with $C_p(E) = 0$. This implies that $u \in W^{1,p}(\mathbf{B})$ has a fine boundary limit $u^*(\xi)$ for p-q.e. $\xi \in \partial\mathbf{B}$ (see [3], [7] and [11]).

Our aim in this note is to show the following theorem.

Theorem 1. *Let φ be a positive nonincreasing function on the interval $(0,\infty)$ satisfying*

$$A^{-1}\varphi(r) \le \varphi(r^2) \le A\varphi(r) \qquad \textit{for every } r > 0 \tag{1}$$

with a positive constant A and

$$\int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr = \infty, \qquad 1 < p < \infty. \tag{2}$$

Let u be a Sobolev function in $W^{1,p}(\mathbf{B})$ such that

$$\int_{U(\varepsilon)} |\nabla u(x)|^p dx \le \varepsilon^p \varphi(\varepsilon) \qquad \textit{for any } \varepsilon > 0, \tag{3}$$

where $U(\varepsilon) = \{x \in \mathbf{B} : |u(x)| < \varepsilon\}$. If there exists $E \subset \partial\mathbf{B}$ such that $C_p(E) > 0$ and the p-precise extension u^* vanishes on E, then $u = 0$.

Remark. P. Koskela [1] has recently treated the case where

$$\begin{aligned}\varphi(r) \quad = \quad & [\log(1/r)]^{p-1}, \qquad [\log(1/r)]^{p-1}[\log\log(1/r)]^{p-1}, \\ & [\log(1/r)]^{p-1}[\log\log(1/r)]^{p-1}[\log\log\log(1/r)]^{p-1}, \cdots\end{aligned}$$

for small r; these functions clearly satisfy (2).

Next we show that our result is sharp in the following sense.

Theorem 2. *Let φ be a positive nonincreasing function on the interval $(0,\infty)$ satisfying (1) and*

$$\int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty, \qquad 1 < p < \infty. \tag{4}$$

Then there exists a continuous function u on R^n such that $u \in W^{1,p}(R^n)$, u satisfies (3), $u > 0$ on $\mathbf{B}$ and $u = 0$ outside $\mathbf{B}$.

2. Proof of Theorem 1

We use the notation $B(x,r)$ to denote the open ball centered at x with radius r; hence $\mathbf{B} = B(0,1)$.

Let us begin with the following lemma, which is an easy consequence of the definition of C_p.

Lemma 1. *Let $1 < p < \infty$. For $f \in L^p(R^n)$, set*

$$E_f = \{x \in R^n : \int_{B(x,1)} |x-y|^{1-n}|f(y)|dy = \infty\}.$$

Then $C_p(E_f) = 0$ (see e.g. [7, Proposition 5.1.1 and Theorem 6.8.2]).

Next we prepare the following technical lemma.

Lemma 2. *Let φ be as in Theorem 1. Then there exists a positive nondecreasing function a on $(0,\infty)$ satisfying* (1) *and such that*

$$\int_0^1 a(r) r^{-1} dr = \infty \tag{5}$$

and

$$\int_0^1 a(r)^p \varphi(r) r^{-1} dr < \infty. \tag{6}$$

In fact, it suffices to consider

$$a(r) = [\varphi(r)]^{-1/(p-1)} \left(\int_r^2 [\varphi(t)]^{-1/(p-1)} t^{-1} dt \right)^{-1}$$

for $r \le 1$; set $a(r) = a(1)$ for $r > 1$.

Throughout this note, let M denote various constants independent of the variables in question.

Now we are ready to prove Theorem 1. For this purpose, we may assume that $u \geqq 0$ a.e. on $\mathbf{B}$. We suppose that $\{x \in \mathbf{B} : u(x) > 0\}$ has positive measure, and obtain a contradiction.

For functions f and g, define

$$f^+(x) = \max\{f(x), 0\}$$

and

$$f \wedge g(x) = \min\{f(x), g(x)\}.$$

For each positive integer j, set

$$u_j(x) = 2^j [(u - 2^{-j})^+ \wedge 2^{-j}]$$

and

$$v = \sum_{j=1}^{\infty} a_j u_j \qquad \text{with } a_j = a(2^{-j})$$

for the function a in Lemma 2. Note that

$$|\nabla u_j| = 2^j |\nabla u| \qquad \text{a.e. on } G_j = \{x : 2^{-j} < u(x) < 2^{-j+1}\}$$

and

$$|\nabla u_j| = 0 \qquad \text{a.e. outside } G_j.$$

Hence it follows from (3), (1) and (6) that

$$\begin{aligned} \int |\nabla v|^p dx &= \sum_{j=1}^{\infty} [a_j 2^j]^p \int_{G_j} |\nabla u|^p dx \\ &\le \sum_{j=1}^{\infty} [a_j 2^j]^p [2^{-j+1}]^p \varphi(2^{-j+1}) \\ &\le M \sum_{j=1}^{\infty} {a_j}^p \varphi(2^{-j}) \\ &\le M \int_0^1 a(r)^p \varphi(r) r^{-1} dr < \infty. \end{aligned}$$

Consider p-precise extensions u^*, $v^* \in W^{1,p}(R^n)$ with compact support such that

$$u = u^* \qquad \text{and} \qquad v = v^*$$

for almost every $x \in \mathbf{B}$. If $\xi \in \partial \mathbf{B}$ and

$$\int |\xi - y|^{1-n} |\nabla u^*(y)| dy < \infty,$$

then $u^*(x + r(\xi - x))$ is absolutely continuous on $(0, \infty)$ for almost every $x \in \mathbf{B}$ (see [4, p.725] and [8]).

If $u^*(x) > 0$, $u^*(\xi) = 0$ and $f(r) = v^*(x + r(\xi - x))$ is absolutely continuous on $[0, 1]$, then the line $L(x, \xi) = \{x + r(\xi - x) : r \in [0, 1]\}$ meets with $G_j = \{x \in \mathbf{B} : 2^{-j} < u^*(x) < 2^{-j+1}\}$ for large j, say $j \geqq j_0$, so that

$$\int_0^1 |f'(r)| dr \geqq \sum_{j \geqq j_0} a_j \geqq M \int_0^{2^{-j_0}} a(r) r^{-1} dr = \infty$$

because of (5). This implies that

$$\int_{\mathbf{B}} |\xi - y|^{1-n} |\nabla v^*(y)| dy = \infty,$$

which gives a contradiction by Lemma 1, since $C_p(E) > 0$ and v^* is p-precise on R^n.

3. Proof of Theorem 2

Suppose

$$\int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty$$

and set

$$r = f(t) \equiv A \int_0^t [\varphi(s)]^{-1/(p-1)} s^{-1} ds$$

for a positive constant A. Note here that if $b > 0$, then

$$s^b \varphi(s) \le M(b) t^b \varphi(t) \qquad \text{whenever } t > s > 0 \tag{7}$$

with a positive constant $M(b)$ (see e.g. [6, (φ5)]).

We now consider the inverse function $t = f^{-1}(r)$ and define

$$u(x) = f^{-1}(1 - |x|)$$

for $x \in \mathbf{B}$; define $u = 0$ outside $\mathbf{B}$. In what follows we determine A for which u satisfies (3).

For small $\varepsilon > 0$, find $\delta > 0$ such that

$$\varepsilon = f^{-1}(\delta).$$

Then we have by (7)

$$\begin{aligned}\int_{\{x\in\mathbf{B}:|u(x)|<\varepsilon\}} |\nabla u(x)|^p dx &\leq M\int_0^\delta |(f^{-1})'(r)|^p dr \\ &= M\int_0^\varepsilon \left|\frac{dr}{dt}\right|^{-(p-1)} dt \\ &= A^{1-p}M\int_0^\varepsilon t^{p-1}\varphi(t)dt \\ &\leq A^{1-p}M\varepsilon^p\varphi(\varepsilon).\end{aligned}$$

Hence it suffices to take A such that $A^{1-p}M = 1$.

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