

**NEW FIXED POINT THEOREMS
FOR A FAMILY OF MAPPINGS AND APPLICATIONS
TO PROBLEMS ON SETS WITH CONVEX SECTIONS**

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ABSTRACT. Some new fixed point theorems for a family of mappings are obtained and applied to problems on sets with convex sections that were first studied by Ky Fan.

1. INTRODUCTION

In this paper we shall employ a partition of unity and Tychonoff's fixed point theorem to prove some new fixed point theorems for a family of mappings defined in product spaces. Our theorems are particularly well suited to treat problems on sets with convex sections. It is well known that the latter have applications in minimax inequalities of von Neumann type and in game theory, for example, the existence of Nash points.

Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space. Write $X = \prod_{i \in I} X_i$ and let $\{A_i\}_{i \in I}$ be a family of subsets of X . Then the problem on sets with convex sections is to find suitable conditions on $\{A_i\}_{i \in I}$ such that the family has a nonempty intersection.

When I is a finite set and X is compact, Ky Fan [Fan] first imposed a convexity and an openness condition on each A_i which assured that the family $\{A_i\}_{i \in I}$ has a nonempty intersection. A new proof of Fan's result was given by Browder [Br] using his fixed point theorem. The idea of Browder was employed in [T] and [T.T] to improve Fan's result, but this fixed point technique cannot be applied to treat the case when I is infinite. The difficulty is that the intersection of infinitely many open sets may not be open.

Fan's result was generalized in two directions. Ma [Ma] employed a partition of unity and an intersection result in [Fan3] (one could also use Tychonoff's fixed point theorem) to generalize Fan's result to the case when I is infinite. Then Fan [Fan2] generalized Ma's result by considering two families of subsets to relax the convexity and openness conditions. Fan's latter result was generalized by Shih and Tan [S.T] to noncompact settings.

On the other hand, Fan [Fan2] generalized his original result by relaxing compactness and convexity conditions and showed that the intersection of some subfamily

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of $\{A_i\}_{i \in I}$ was not empty (see Theorem 15 in [Fan2]). The main idea is to change the question into an intersection problem for KKM maps. Fan's idea was then employed by Shih and Tan [S.T] to improve Fan's result to two families of subsets with a different compactness condition. However, this method cannot be applied to treat the case when I is infinite due to the fact that the union of infinitely many closed sets may not be closed. Hence, when I is an infinite set, the problem has not been solved until now.

In this paper we shall first consider the problem of existence of fixed points for a family of mappings $\{\psi_i\}_{i \in I}$, where $\psi_i : X^i \rightarrow 2^{X^i}$, that is, to find $x \in X$ such that $x_i \in \psi_i(x^i)$ for all $i \in I$. (The precise meanings of these symbols will be given later). Since the space X involved in our fixed point theorems may not be compact, we employ a special technique in the proof of our main fixed point theorem to utilize a partition of unity and Tychonoff's fixed point theorem.

As applications of our new fixed point theorems, we deduce new results on sets with convex sections. We show that all the above generalizations of Fan's result on sets with convex sections can be unified in a more general result which applies whether the index set I is finite or infinite. Furthermore, all of the previous compactness conditions used in [Fan2] and [S.T] are replaced by a weaker compactness condition. Hence, our results on sets with convex sections not only unify the previous results but also are very general.

2. FIXED POINT THEOREMS AND RESULTS ON SETS WITH CONVEX SECTIONS

Throughout this section, let I be an index set and for each $i \in I$ let E_i be a Hausdorff topological vector space. Let X_i be a nonempty convex subset of E_i , $X = \prod_{i \in I} X_i$ and $X^i = \prod_{j \neq i, j \in I} X_j$. We write $X = X_i \otimes X^i$. Then, for each fixed $i \in I$ and each $x \in X$, we write $x = (x_i, x^i)$, where $x_i \in X_i$ and $x^i \in X^i$.

Assume that E is a Hausdorff topological vector space and Y is a nonempty convex subset in E . We denote by 2^Y the family of all subsets of Y and by $\text{co}B$ the convex hull of a set B . Let $G : Y \rightarrow 2^Y$ be a multivalued map. We define G^{-1} , $G^* : Y \rightarrow 2^Y$ by $G^{-1}(y) = \{x \in Y : y \in G(x)\}$ and $G^*(y) = \{x \in Y : y \notin G(x)\}$, respectively. Some properties of the above maps can be found in Lemma 3.2 of [Lan]. We shall directly apply these properties.

We start with the following new fixed point theorem for a family of mappings defined in product spaces, which is a key result in this paper.

Theorem 2.1. *For each $i \in I$, let $\psi_i : X^i \rightarrow 2^{X^i}$ be a map. Assume that the following conditions hold.*

- (h₁) *For each $x^i \in X^i$, $\psi_i(x^i) \neq \emptyset$.*
- (h₂) *For each $x^i \in X^i$, $\psi_i(x^i)$ is convex.*
- (h₃) *For each $i \in I$ and each $y_i \in X_i$, $\psi_i^{-1}(y_i)$ is open in X^i .*
- (h₄) *If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that $X_i^0 \cap \psi_i(x^i) \neq \emptyset$ for each $x^i \in X^i \setminus D(i)$.*

Then there exists $x \in X$ such that $x_i \in \psi_i(x^i)$ for all $i \in I$.

Proof. For each $i \in I$, we define a map $\phi_i : X_i \rightarrow 2^{X^i}$ by $\phi_i(x_i) = \psi_i^*(x_i)$. Then ϕ_i satisfies the following conditions.

- (a) For each $x_i \in X_i$, $\phi_i(x_i)$ is closed in X^i by (h₃).
- (b) For each $i \in I$, $\bigcap_{x_i \in X_i^0} \phi_i(x_i)$ is compact in X^i .

In fact, if X^i is compact, $\bigcap_{x_i \in X_i^0} \phi_i(x_i)$ is compact since $\bigcap_{x_i \in X_i^0} \phi_i(x_i)$ is closed in X^i by (a). If X^i is not compact, $\bigcap_{x_i \in X_i^0} \phi_i(x_i) \subset D(i)$ by (h_4) and thus is compact.

(c) For each $i \in I$, $\bigcap_{x_i \in X_i} \phi_i(x_i) = \emptyset$ by (h_1) .

Now, we prove that there exist $a_{i1}, \dots, a_{il_i} \in X_i$ such that

$$(2.1) \quad \bigcap_{x_i \in X_i^0} \phi_i(x_i) \cap \left(\bigcap_{k=1}^{l_i} \phi_i(a_{ik}) \right) = \emptyset.$$

In fact, if not, then for every finite set $\{y_1, \dots, y_n\} \subset X_i$,

$$\bigcap_{x_i \in X_i^0} \phi_i(x_i) \cap \left(\bigcap_{j=1}^n \phi_i(y_j) \right) \neq \emptyset.$$

Let $G(y) = \bigcap_{x_i \in X_i^0} \phi_i(x_i) \cap \phi_i(y)$ for $y \in X_i$. Then the family $\{G(y) : y \in X_i\}$ has the finite intersection property. Note that $G(y)$ is compact for each $y \in X_i$ by (a) and (b). It follows that $\bigcap_{y \in X_i} G(y) \neq \emptyset$, and thus $\bigcap_{y \in X_i} \phi_i(y) \neq \emptyset$, which contradicts (c).

By (2.1) we have

$$(2.2) \quad \bigcup_{x_i \in X_i^0} \psi_i^{-1}(x_i) \cup \left(\bigcup_{k=1}^{l_i} \psi_i^{-1}(a_{ik}) \right) = X^i.$$

Let $F_i = \text{co}(X_i^0 \cup \{a_{i1}, \dots, a_{il_i}\})$. Then F_i is compact in X_i . For each $i \in I$, let $F^i = \prod_{j \neq i, j \in I} F_j$. Then F^i is a compact subset of X^i for each $i \in I$. By (2.2) we have

$$(2.3) \quad F^i \subset \bigcup_{x_i \in X_i^0} \psi_i^{-1}(x_i) \cup \left(\bigcup_{k=1}^{l_i} \psi_i^{-1}(a_{ik}) \right).$$

Since F^i is compact, there exist $b_{i1}, \dots, b_{it_i} \in X_i$ such that

$$(2.4) \quad F^i \subset \bigcup_{j=1}^{t_i} \psi_i^{-1}(b_{ij}) \cup \left(\bigcup_{k=1}^{l_i} \psi_i^{-1}(a_{ik}) \right).$$

Let $\{c_{i1}, \dots, c_{in_i}\} = \{a_{i1}, \dots, a_{il_i}, b_{i1}, \dots, b_{it_i}\}$. We rewrite (2.4) as follows.

$$(2.5) \quad F^i \subset \bigcup_{k=1}^{n_i} \psi_i^{-1}(c_{ik}).$$

Let $K_i = \text{co}\{c_{i1}, \dots, c_{in_i}\}$, $K = \prod_{i \in I} K_i$ and $K^i = \prod_{j \neq i, j \in I} K_j$. We write $K = K_i \otimes K^i$. Denote by S_i the vector subspace of E_i generated by K_i . Then S_i is locally convex since it is a finite dimensional subspace. We note that K and K^i are compact in $\prod_{i \in I} S_i$ and $\prod_{j \in I, j \neq i} S_j$, respectively, and $K^i \subset F^i$ for each $i \in I$. For each $i \in I$, let $O_{ik} = \psi_i^{-1}(c_{ik}) \cap K^i$ for each $k \in \{1, \dots, n_i\}$. Then O_{ik} is open in K^i for each $k \in \{1, \dots, n_i\}$, and $K^i = \bigcup_{k=1}^{n_i} O_{ik}$.

Let $\{f_{i1}, \dots, f_{in_i}\}$ be a non-negative continuous partition of unity subordinate to this open covering of K^i . Then

$$f_{ik}(x^i) = 0 \quad \text{for} \quad x^i \in K^i \setminus O_{ik} \quad \text{and} \quad \sum_{k=1}^{n_i} f_{ik}(x^i) = 1 \quad \text{for} \quad x^i \in K^i.$$

Define a continuous map $g_i : K^i \rightarrow 2^{K^i}$ by $g_i(x^i) = \sum_{k=1}^{n_i} f_{ik}(x^i)c_{ik}$. Note that $f_{ik}(x^i) \neq 0$ implies $x^i \in O_{ik}$. This implies $x^i \in \psi_i^{-1}(c_{ik})$, and thus $c_{ik} \in \psi_i(x^i)$. It follows from (h_2) that $g_i(x^i) \in \psi_i(x^i)$.

Define a map $f : K \rightarrow K$ by $f(x) = (g_i(x^i))_{i \in I}$. Since for each $x \in K$ we have $x^i \in K^i$ and $g_i(x^i) \in K^i$, it follows that f is well-defined and continuous. By Tychonoff's fixed point theorem [Ty], f has a fixed point $x = (x_i)_{i \in I} \in K$. This implies $x_i \in g_i(x^i)$ for all $i \in I$. Since $g_i(x^i) \subset \psi_i(x^i)$ for each $i \in I$, we have $x_i \in \psi_i(x^i)$ for all $i \in I$. \square

Remark 2.1. The condition (h_4) is equivalent to the following condition.

(C_1) If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that for each $x^i \in X^i \setminus D(i)$ there exists $x_i^0 \in X_i^0$ such that $x_i^0 \in \psi_i(x^i)$.

The following conditions are special cases of (h_4) .

(C_2) If X is not compact, assume that there exist a nonempty compact convex subset K of X and a nonempty compact subset D of X such that for each $x \in X \setminus D$ there exists $y \in K$ such that $y_i \in \psi_i(x^i)$ for all $i \in I$.

To see that (C_2) implies (h_4) , we show that (C_2) implies (C_1) . We define maps $p_i : X \rightarrow X_i$ and $p^i : X \rightarrow X^i$ by $p_i(x) = x_i$ and $p^i(x) = x^i$. Let $D(i) = P^i(D)$ and $X_i^0 = p_i(K)$. Now, for each $i \in I$ and each $x^i \in X^i \setminus D(i)$ (i.e., $x^i \notin D(i)$) and for any fixed $x_i \in X_i$, we have $x = (x_i, x^i) \notin D$. By (C_2) there exists $y \in K$ such that $y_i \in \psi_i(x^i)$, that is, $y_i \in X_i^0$ and $y_i \in \psi_i(x^i)$. Hence (C_1) holds.

(C_3) If X^i is not compact, assume that there exists a nonempty compact subset $D(i)$ of X^i such that $\bigcap_{x^i \in X^i \setminus D(i)} \psi_i(x^i) \neq \emptyset$.

(C_4) If X^i is not compact, assume that there exist $z_i \in X_i$ and a nonempty compact subset $D(i)$ of X^i such that $\psi_i^*(z_i) \subset D(i)$.

(C_5) If X is not compact, assume that there exists a nonempty compact subset $D \subset X$ such that $D \cap \prod_{i \in I} \psi_i(y^i) \neq \emptyset$ for $y \in X \setminus D$.

In fact, (C_3) and (C_4) are equivalent, and (C_5) implies (C_4) .

The following result is a slight generalization of Theorem 2.1.

Theorem 2.2. *For each $i \in I$, let $\phi_i : X^i \rightarrow 2^{X^i}$ be a map. Assume that the following conditions hold.*

(H_1) *For each $x^i \in X^i$, $\phi_i(x^i) \neq \emptyset$.*

(H_2) *For each $i \in I$ and each $y_i \in X_i$, $\phi_i^{-1}(y_i)$ is open in X^i .*

(H_3) *If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that $X_i^0 \cap \text{co } \phi_i(x^i) \neq \emptyset$ for each $x^i \in X^i \setminus D(i)$.*

Then there exists $x \in X$ such that $x_i \in \text{co } \phi_i(x^i)$ for all $i \in I$.

Proof. For each $i \in I$, we define a map $\psi_i : X^i \rightarrow 2^{X^i}$ by $\psi_i(x^i) = \text{co } \phi_i(x^i)$. Then it is easy to verify that ψ_i satisfies (h_1) , (h_2) and (h_4) . By Lemma 5.1 in [Y.P] we see that (H_2) implies ψ_i satisfies (h_3) . The result follows from Theorem 2.1. \square

By Theorem 2.2 we obtain the following result on sets with convex sections.

Theorem 2.3. *Let $\{A_i\}_{i \in I}$ be a family of subsets of X such that the following conditions hold.*

(S_1) *For each $i \in I$ and each $x^i \in X^i$, $\{x_i \in X_i : (x_i, x^i) \in A_i\} \neq \emptyset$.*

(S_2) *For each $i \in I$ and each $x_i \in X_i$, the section $\{x^i \in X^i : (x_i, x^i) \in A_i\}$ is open in X^i .*

(S₃) If X^i is not compact, assume that there exist a nonempty compact convex subset X_i^0 of X_i and a nonempty compact subset $D(i)$ of X^i such that for each $x^i \in X^i \setminus D(i)$, $X_i^0 \cap \text{co}\{x_i \in X_i : (x_i, x^i) \in A_i\} \neq \emptyset$.

Then there exists $x \in X$ such that $x_i \in \text{co}\{x_i \in X_i : (x_i, x^i) \in A_i\}$ for all $i \in I$.

Proof. For each $i \in I$, we define a map $\phi_i : X^i \rightarrow 2^{X_i}$ by

$$\phi_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in A_i\}.$$

Then it is easy to verify that $\{\phi_i\}_{i \in I}$ satisfies all the condition of Theorem 2.2. The result follows. \square

Remark 2.2. The following conditions (C'_2) and (C'_5) are special cases of (S_3) .

(C'_2) If X is not compact, assume that there exist a nonempty compact convex subset K of X and a nonempty compact subset D of X such that for each $x \in X \setminus D$ there exists $y \in K$ such that $(y, x^i) \in A_i$ for all $i \in I$.

We show (C'_2) implies (S_3) . Assume that (C'_2) holds. For each $i \in I$, let $\psi_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in A_i\}$, and let $D(i)$ and X_i^0 be as in the proof of (C_2) implies (C_1) . Then $\{\psi_i\}_{i \in I}$ satisfies (C_2) . Since (C_2) implies (C_1) and (C_1) implies (h_4) , we have $X_i^0 \cap \psi_i(x^i) \neq \emptyset$ for each $x^i \in X^i \setminus D(i)$; that is, $X_i^0 \cap \{x_i \in X_i : (x_i, x^i) \in A_i\} \neq \emptyset$ for each $x^i \in X^i \setminus D(i)$. This implies (S_3) . \square

(C'_5) If X is not compact, assume that there exists a nonempty compact subset $D \subset X$ such that $D \cap \prod_{i \in I} \psi_i(y^i) \neq \emptyset$ for $y \in X \setminus D$, where $\psi_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in A_i\}$.

Using Theorem 2.3, we obtain the following general result on sets with convex sections for two families of subsets.

Theorem 2.4. *Assume that all the conditions of Theorem 2.3 hold. Let $\{B_i\}_{i \in I}$ be a family of subsets of X such that the following condition holds.*

(H) *For each $x \in X$ there exists a subset $I(x) \subset I$ such that for $i \in I(x)$,*

$$\text{co}\{y_i \in X_i : (y_i, x^i) \in A_i\} \subset \{y_i \in X_i : (y_i, x^i) \in B_i\}.$$

Then there exists $x \in X$ such that $\bigcap_{i \in I(x)} B_i \neq \emptyset$.

Proof. There exists $x \in X$ such that $x_i \in \text{co}\{x_i \in X_i : (x_i, x^i) \in A_i\}$ for all $i \in I$ by Theorem 2.3. Hence, for this x , $x_i \in \{y_i \in X_i : (y_i, x^i) \in B_i\}$ for $i \in I(x)$. This implies $x \in B_i$ for all $i \in I(x)$. \square

Remark 2.3. Theorem 2.4 generalizes Theorem 15 in [Fan2] and Theorem 5 in [S.T] in the following ways: (i) the set I may be infinite; (ii) the conditions (S_3) are weaker than the condition (c) in Theorem 15 of [Fan2] (see (C'_2) in Remark 2.2) and the conditions (d'') of Theorem 5 in [S.T] (see (C'_5) in Remark 2.2); and (iii) the condition (H) is weaker than the condition (d) in Theorem 15 of [Fan2] and the condition (c) in Theorem 5 of [S.T]. Moreover, we see that our method is different from those used in Theorem 15 of [Fan2] and Theorem 5 in [S.T].

Even when $I(x) = I$ for $x \in X$, Theorem 2.4 generalizes Theorem 2 in [S.T]; and when $A_i = B_i$, Theorem 2.4 generalizes Theorem 16 in [Fan2] and those noted in [Fan2], namely generalizations of Theorem 2 in [Ma] and Theorem 1 in [Fan1].

We can state an analytic formulation of Theorem 2.4, but we only give the following simple result.

Theorem 2.5. *Let $\{f_i\}_{i \in I} : X \rightarrow \mathbb{R}$ be a family of functions and $\{t_i\}_{i \in I}$ a sequence of real numbers. Assume that the following conditions hold.*

- (s₁) *For each $i \in I$ and each $x^i \in X^i$, $f_i(\cdot, x^i)$ is a quasiconcave function on X_i .*
- (s₂) *For each $i \in I$ and each $x_i \in X_i$, $f_i(x_i, \cdot)$ is a lower semicontinuous function on X^i .*
- (s₃) *For each $i \in I$ and each $x^i \in X^i$, there exists $y_i \in X_i$ such that $f_i(y_i, x^i) > t_i$.*
- (s₄) *If X is not compact, assume that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset D of X such that for each $x \in X \setminus D$, there exists $y \in X_0$ such that $f_i(y_i, x^i) > t_i$ for all $i \in I$.*

Then there exists $x \in X$ such that $f_i(x) > t_i$ for all $i \in I$.

Proof. For each $i \in I$, we define $\psi_i : X^i \rightarrow 2^{X_i}$ by $\psi_i(x^i) = \{x_i \in X_i : f_i(x_i, x^i) > t_i\}$. Then it is easy to verify that $\{\psi_i\}_{i \in I}$ satisfies all the conditions of Theorem 2.1. The result follows. \square

The above result generalizes Theorem 3 in [Fan1].

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