PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 1133–1138
S 0002-9939(98)04394-9

DERIVATIONS IMPLEMENTED BY LOCAL MULTIPLIERS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. A condition on a derivation of an arbitrary C^*-algebra is presented entailing that it is implemented as an inner derivation by a local multiplier.

It is an outstanding open question whether every derivation of a C^*-algebra A can be implemented as an inner derivation by a local multiplier, that is, an element in the direct limit of the multiplier algebras of the closed essential ideals of A. An affirmative answer was given by Elliott [4] for AF-algebras, and by Pedersen [11] for general separable C^*-algebras. In fact, it suffices to assume that every closed essential ideal of A is σ-unital; hence Pedersen's result entails Sakai's theorem that every derivation of a simple unital C^*-algebra is inner. But only an affirmative answer in the non-separable case would cover, extend and unify the results that every derivation of a simple C^*-algebra is inner in the multiplier algebra [13] and that all derivations of von Neumann algebras [6], [12] and AW^*-algebras [10] are inner. This quest becomes even more attractive by the recent results in [9] and [14] implying that, if a derivation δ on A is inner in the multiplier algebra, then there is a *local* multiplier a of A implementing δ such that $\|\delta\| = 2\,\|a\|$.

No progress on the above question seems to have been made since it was raised in [11] (see also [4]). The purpose of this note is to present a criterion on a given derivation δ of a (possibly non-separable) C^*-algebra A implying that δ is inner in the local multiplier algebra $M_{\mathrm{loc}}(A)$. Though this criterion, inspired by Herstein's work [5], is rather algebraic in nature, it is hoped that some approximate version may eventually yield a positive solution of the general problem.

1. NOTATION AND PRELIMINARIES

Throughout this paper, $M(A)$ will denote the multiplier algebra of the C^*-algebra A. A left ideal L of A is said to be *essential* if its left annihilator $L^\perp = \{a \in A \mid aL = 0\}$ is zero. For a (closed) two-sided ideal I, the left annihilator coincides with the right annihilator, and $I + I^\perp$ is a (closed) essential ideal. Given two closed essential ideals I, J in A such that $J \subseteq I$, J is an essential ideal in $M(I)$ and hence $M(I)$ embeds isometrically into $M(J)$. Forming the C^*-direct limit of the directed family of multiplier algebras so obtained yields *the local multiplier algebra* of A,

Received by the editors September 23, 1996.
1991 *Mathematics Subject Classification.* Primary 46L57; Secondary 47B47, 16N60.
Key words and phrases. C^*-algebras, derivations, local multipliers.
This work was done while the author was a Visiting Fellow at *The Fields Institute for Research in Mathematical Sciences*, Waterloo, Ontario, Canada, supported by the Deutsche Forschungsgemeinschaft (DFG), to both of which he is very grateful.

denoted by $M_{\mathrm{loc}}(A)$. If we merely take the algebraic direct limit, we obtain a dense *-subalgebra of $M_{\mathrm{loc}}(A)$ which is called *the bounded symmetric algebra of quotients*, $Q_b(A)$ of A. The reason for this terminology is that $Q_b(A)$ is the bounded part of the purely algebraic version, *the symmetric ring of quotients* $Q_s(A)$ of A in the sense of Kharchenko, where A is considered as a semiprime ring only. For more details on $Q_s(A)$ we refer to [7]. Another important interrelation between $Q_s(A)$ and $Q_b(A)$ is noted in [1]: every element $q \in Q_s(A)$ can be written as $c^{-1}q_0$, where $q_0 \in Q_b(A)$, c belongs to C_b, the center of $Q_b(A)$, and is not a divisor of zero. The commutative *-algebra C_b is dense in the center of $M_{\mathrm{loc}}(A)$ [2] and is the bounded part of the center C of $Q_s(A)$; thus it is called *the bounded extended centroid* of A. Whenever J is an ideal of A, there is a unique projection $\mathrm{c}(J)$ in C_b such that the annihilator of JC in AC is $(1-\mathrm{c}(J))AC$; we call $\mathrm{c}(J)$ *the central support* of J. If $x \in A$ then $e_x := \mathrm{c}(AxA)$ is the central support of x (which is in fact the central support projection of x within the AW^*-algebra $Z(M_{\mathrm{loc}}(A))$. Whenever $a, b \in M(A)$, we shall denote by $M_{a,b}$ the two-sided multiplication $x \mapsto axb$ on A, and by δ_a the inner derivation $x \mapsto xa - ax$.

It has emerged that, in working with local multipliers, it is often rather expedient and sometimes inevitable to also appeal to the surrounding algebraic framework, that is, to work within $Q_s(A)$ instead of $Q_b(A)$ only. The reason is the following. There is no way of making a non-invertible element of a C^*-algebra A invertible by enlarging A to a bigger C^*-algebra, but in $Q_s(A)$ such an element may become invertible, and hence many more equations can be solved within the non-C^*-algebra $Q_s(A)$. At the end, an additional argument is then needed to finally find the solution (to the original problem) within the C^*-algebraic frame, that is, $M_{\mathrm{loc}}(A)$. Thus, working with local multipliers typically divides into two steps, a first purely algebraic one and a second, entirely independent analytic argument. This route is very well illustrated in [3], and we shall follow it subsequently again.

2. The results

The analytic step in our arguments is provided by the following observation.

Lemma. *Let L be an essential left ideal in a C^*-algebra A. Let $f\colon J \to A$ be a linear mapping defined on a subspace J of A. If, for some derivation $\delta\colon A \to A$, the identity*

$$f(x)u = -x\delta(u) \qquad (x \in J,\ u \in L)$$

holds, then f is bounded with norm at most $\|\delta\|$.

Proof. Let π be an irreducible representation of A. By hypothesis,

$$\pi(f(x)y)\pi(z)\pi(u) = -\pi(x)\delta_\pi(\pi(yzu))$$

for all $x \in J$, $y, z \in A$ and $u \in L$, where δ_π denotes the induced derivation on $\pi(A)$. Hence,

$$\begin{aligned}\left\|M_{\pi(f(x)y),\pi(u)}\,\pi(z)\right\| &\le \|\pi(x)\|\,\|\delta_\pi\|\,\|\pi(y)\|\,\|\pi(z)\|\,\|\pi(u)\| \\ &\le \|x\|\,\|\delta\|\,\|y\|\,\|\pi(z)\|\,\|\pi(u)\|,\end{aligned}$$

wherefore

$$\left\|M_{\pi(f(x)y),\pi(u)}\right\| \le \|x\|\,\|\delta\|\,\|y\|\,\|\,\|\pi(u)\|$$

for all $x \in J$, $y \in A$ and $u \in L$. Let I be the closed ideal $\overline{LL^*}$. If $\ker\pi$ does not contain I, there is $u \in L$ such that $\pi(u) \neq 0$. By [8, Proposition 2.3], $\|M_{\pi(f(x)y),\pi(u)}\| = \|\pi(f(x)y)\|\,\|\pi(u)\|$, whence the above inequality entails that

$$\|\pi(f(x)y)\| \leq \|x\|\,\|\delta\|\,\|y\|.$$

Since each irreducible representation of I extends to an irreducible representation of A not vanishing on I, it follows that $\|f(x)y\| \leq \|x\|\,\|\delta\|\,\|y\|$ for all $x \in J$ and $y \in I$. Since I is essential (as L is essential), we conclude that

$$\|f(x)\| = \sup\{\|f(x)y\| \mid y \in I,\ \|y\| \leq 1\} \leq \|\delta\|\,\|x\|$$

for all $x \in J$, as required. □

We shall apply this lemma below to show that a certain derivation that is inner when extended to $Q_s(A)$, is in fact inner in $M_{\mathrm{loc}}(A)$. The most general result on innerness of derivations in the local multiplier algebra so far has been Pedersen's result [11, Proposition 2]. (We use this occasion to note that one of the assertions in [11, Lemma 1], viz. the absolute summability of $(y_n)_{n\in\mathbf{N}}$, is not proved and in fact cannot be proven, as simple counterexamples show. Fortunately, this does not interfere with the subsequent applications of [11, Lemma 1].) Pedersen's condition is on the algebra (A has to be separable), whereas our condition is on the derivation itself. Possibly a synthesis of weakened versions of both may result in the solution of the general question.

Theorem A. *Let δ be a derivation of a C*-algebra A. Suppose there exist an essential left ideal L of A and an element $a \in A$ satisfying $a\delta L = 0$ and $(1-e_a)\delta L = 0$. Then there is $h \in Q_b(A)$ such that*

$$\delta = \delta_h,\ ah = 0,\ Lh = 0,\ \textit{and}\ \|h\| \leq \|\delta\|.$$

Proof. For all $u \in L$ and $y \in A$ we have

$$ay\delta u + a(\delta y)u = a\delta(yu) = 0$$

by assumption, whence

$$(1) \qquad M_{a,\delta u} + M_{a,u} \circ \delta = 0 \qquad (u \in L).$$

On the ideal $J = AaA$ we define $f\colon J \to A$ by $\sum_i x_i a y_i \mapsto \sum_i x_i a\delta y_i$ whenever x_i, y_i are finitely many elements in A. Note that, by (1),

$$\sum_i x_i a(\delta y_i)u = -\sum_i x_i a y_i \delta u,$$

whence

$$(2) \qquad f(x)u = -x\delta u \qquad (x \in J,\ u \in L)$$

and

$$(3) \qquad f(x)yu = -x\delta(yu) \qquad (x \in J,\ y \in A,\ u \in L).$$

By (2), $(f(x_1 + \lambda x_2) - f(x_1) - \lambda f(x_2))u = 0$ for all $x_1, x_2 \in J$, $\lambda \in \mathbf{C}$ and $u \in L$, whereas $x = 0$ implies that $f(x)u = 0$ for all $u \in L$. Since L is essential, it follows that f is a well-defined linear mapping on J.

Applying the Lemma to (2), we conclude that f is bounded with norm at most $\|\delta\|$. Hence, replacing J by its closure, we may assume that J is closed.

Let $J^\perp$ denote the annihilator of J in A. If $x_1 \in J$, $x_2 \in J^\perp$, we put $\tilde{f}(x_1+x_2) = f(x_1)$. Then, as $(1-e_a)\delta L = 0$,

$$\tilde{f}(x_1+x_2)u = f(x_1)u = -x_1\delta u = -(x_1+x_2)e_a\delta u = -(x_1+x_2)\delta u \qquad (u \in L).$$

Hence, replacing J by $J+J^\perp$ and f by $\tilde{f}$, we may assume that J is an essential closed ideal in A.

By (2),

$$(f(yx) - yf(x))u = -(yx-yx)\delta u = 0 \qquad (x \in J,\ y \in A,\ u \in L),$$

whence f is a left A-module map. Put $g = f - \delta$. Then,

$$\begin{aligned} g(xy)u &= f(xy)u - \delta(xy)u = -xy\delta u - \delta(xy)u \\ &= -\delta(xyu) \\ &= -(\delta x)yu - x\delta(yu) = f(x)yu - (\delta x)yu \\ &= g(x)yu \end{aligned}$$

for all $x \in J$, $y \in A$ and $u \in L$ so that g is a right A-module map from J into A. Moreover, if $x, y \in J$, then, by (3),

$$f(x)yu = -x\delta(yu) = -xy\delta u - x(\delta y)u = x(f(y) - \delta y)u = xg(y)u \qquad (u \in L),$$

and thus $f(x)y = xg(y)$. As a result, (f, g) is a double centralizer of J represented by an element $h \in M(J)$. By definition, $\delta = f - g = R_h - L_h = \delta_h$ on J. From this we infer that

$$\begin{aligned} (\delta y)x &= \delta(yx) - y(\delta x) \\ &= f(yx) - g(yx) - yf(x) + yg(x) \\ &= yg(x) - g(yx) \\ &= yhx - hyx = [y, h]x \end{aligned}$$

for all $x \in J$ and $y \in A$. Since J is essential, this yields that $\delta = \delta_h$ on A.

The identity

$$a(yuh - hyu) = a\delta(yu) = 0 \qquad (y \in A, u \in L)$$

implies that

$$M_{a,uh} = M_{ah,u} \qquad (u \in L). \tag{4}$$

Therefore, the mapping

$$\sum_i x_i a y_i + v \mapsto \sum_i x_i a h y_i \qquad (x_i, y_i \in A,\ v \in (AaA)^\perp)$$

is a well-defined A-bimodule map from the essential ideal $AaA + (AaA)^\perp$ into A which gives rise to an element $\lambda \in C$ with the property $\lambda a = ah$. This together with (4) entails that

$$M_{a,uh-\lambda u} = M_{a,uh} - \lambda\, M_{a,u} = 0 \qquad (u \in L),$$

whence $0 = e_a u(h-\lambda) = u(h-\lambda)$ as $e_a h = h$ and $e_a\lambda = \lambda$. Replacing h by $h - \lambda$, we thus obtain $\delta = \delta_h$ as well as $ah = 0$ and $Lh = 0$. In particular, $xhu = -x\delta u$ for all x in the domain of h and $u \in L$ (that is, (2)); thus the same reasoning as before shows that h still is bounded with $\|h\| \le \|\delta\|$. $\square$

A more symmetric version of the condition appearing in Theorem A is obtained in our first corollary.

Corollary 1. *Let δ be a derivation on a C*-algebra A. For each pair of elements $a, b \in A$ such that*

$$M_{a,\delta b} + M_{a,b} \circ \delta = 0$$

there is $h \in Q_b(A)$ with the properties

$$e_a e_b\, \delta = \delta_h,\ \ ah = bh = 0 \ \ \text{and}\ \|h\| \le \|\delta\|.$$

Proof. We first follow the proof of Theorem A. Put $J = AaA$ and $L = Ab$. Note that the left annihilators of L and AbA coincide. By assumption,

$$ay\delta(xb) + a(\delta y)xb = ay(\delta x)b + ayx\delta b + a\delta(yx)b - ay(\delta x)b = 0$$

for all $x, y \in A$. Therefore (1) holds for all $u \in L$. Defining $f\colon J \to A$ as above, we thus obtain a well-defined bounded left A-module map $e_b f$ which we may extend to $\bar{J} + J^{\perp}$ as before. Note that (2) changes to

$$\text{(2')} \qquad e_b f(x) e_a u = -x\, \delta(e_a e_b u) \qquad (x \in \bar{J} + J^{\perp},\ u \in L).$$

Letting $e_b g = e_b f - e_b \delta$, we obtain a bounded right A-module map on $\bar{J} + J^{\perp}$ such that $e_b f(x) y = e_b x g(y)$ for all $x, y \in J$. Let h be the element in $Q_b(A)$ corresponding to the local double centralizer $(e_b f, e_b g)$ of A. Then, $e_b \delta = \delta_h$ on J. As above, this entails that $(e_b\delta - \delta_h)A \subseteq J^{\perp}$, so that $e_a e_b \delta = \delta_{h'}$ with $h' = e_a h$. Since we still have (4) with h' instead of h, we find $\lambda' = \lambda' e_a \in C$ such that $\lambda' a = ah'$ as well as $\lambda' u = uh'$ for all $u \in L$. Now define h anew by $h = h' - e_b \lambda'$. Then, $e_a e_b h = h$, $ah = 0$, $Lh = 0$ and $e_a e_b \delta = \delta_h$. A final application of the Lemma yields $\|h\| \le \|\delta\|$. □

Corollary 2. *For every derivation δ on a prime C*-algebra A the following conditions are equivalent.*

(a) *There are a non-zero element $a \in A$ and a non-zero left ideal L of A such that $a\delta L = 0$.*
(b) *There is an element $h \in Q_b(A)$ such that $\delta = \delta_h$ and $yh = 0$ for some non-zero $y \in Q_b(A)$.*

Proof. (a) ⇒ (b) As L is essential and $e_a = 1$, the assertion follows immediately from Theorem A.

(b) ⇒ (a) Let I be a non-zero closed ideal of A such that $y \in M(I)$, and put $L = Iy$. Then, for each non-zero $a \in L$ and all $u \in L$ we have $a\delta u = auh - ahu = 0$, i.e. $a\delta L = 0$. □

Our arrangement of the proof of Theorem A reveals that its algebraic part carries over verbatim to the setting of semiprime rings. Suppose that the element a in Theorem B below is zero. Then $\delta L = 0$, wherefore $(\delta y)u = \delta(yu) - y\delta u = 0$ for all $y \in R, u \in L$ implies that $\delta y = 0$ for all y. Thus, $a = 0$ entails that $\delta = 0$ and Theorem B extends the corresponding statement for prime rings in [5, Theorem], replacing the Martindale ring of quotients by its symmetric version.

Theorem B. *Let δ be a derivation of a semiprime ring R. Suppose there exist an essential left ideal L of R and an element $a \in R$ satisfying $a\delta L = 0$ and $(1 - e_a)\delta L = 0$. Then there is $h \in Q_s(R)$ such that $\delta = \delta_h$, $ah = 0$, and $Lh = 0$.*

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