

## WHEN MUST PURE EXTENSIONS OF COUNTABLE ABELIAN GROUPS NECESSARILY SPLIT?

H. PATRICK GOETERS AND PATRICK KEEF

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ABSTRACT. Necessary and sufficient conditions are given for the group of pure extensions of a countable abelian group by a countable abelian group to equal zero.

### INTRODUCTION

In this work, by the term “group” we will mean “abelian group”. Any notation and terminology not specifically defined can be found in [1].

If  $A$  and  $B$  are groups, then a short exact sequence  $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$  is *pure* if for every integer  $n$ ,  $nB = B \cap nX$ . The collection of pure extensions is denoted by  $\text{Pext}(A, B)$ . This can be shown to agree with the first Ulm subgroup,  $\text{Ext}(A, B)^1 = \bigcap n \text{Ext}(A, B)$ . Many different properties of this bifunctor are established in Chapter IX of [1], and we will assume the results contained there.

We will be interested in describing when  $\text{Pext}(A, B) = 0$ , where  $A$  and  $B$  are countable groups. This question was suggested to the first author by C. Schochet, who noted its importance to subjects of  $KK$ -theory (see [2]). In particular, the graded Kasparov group  $KK_*(M, N)$  of certain  $C^*$ -algebras  $M$  and  $N$  possesses a direct factor of the form  $\text{Pext}(K_*(M), K_*(N))$ . It is important to know when this latter group vanishes, and it should also be noted that every countable abelian group is  $K_0(M)$  for some separable  $C^*$ -algebra  $M$ . In addition, as also noted in [2], by results of Jensen, if  $M = \varinjlim M_i$ , then

$$\text{Pext}(M, N) = \varprojlim^1 \text{Hom}(M_i, N),$$

so that the collection of pure extensions is of some importance to topologists.

Our main result is Theorem 1, where the above problem is completely solved. It is perhaps surprising that the question of when  $\text{Pext}(A, B)$  vanishes can be separated into two distinct cases; when  $B$  is torsion and when it is torsion-free (Corollary 2). In addition, we show that whenever  $A$  and  $B$  are countable, then  $\text{Pext}(A, B)$  is either 0 or has cardinality  $c$  (Corollary 3).

If  $A$  is a group, then  $tA$  will denote the torsion subgroup of  $A$  and  $fA \stackrel{\text{def}}{=} A/tA$ . More generally, if  $\mathcal{P}$  is a collection of primes, then  $t_{\mathcal{P}}A$  will consist of those elements of  $A$  whose order is a product of elements of  $\mathcal{P}$ , and  $t_{\mathcal{P}}^*A$  will consist of those elements whose order is not divisible by any element of  $\mathcal{P}$ . We let  $f_{\mathcal{P}}A = A/t_{\mathcal{P}}A$

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and  $f_{\mathcal{P}}^*A = A/t_{\mathcal{P}}^*A$ . The integers localized at  $\mathcal{P}$  will be denoted by  $\mathbf{Z}_{(\mathcal{P})}$ . A group  $A$  will be said to be  $\mathcal{P}$ -free if  $A_{(\mathcal{P})} \stackrel{\text{def}}{=} A \otimes \mathbf{Z}_{(\mathcal{P})}$  is a free  $\mathbf{Z}_{(\mathcal{P})}$ -module, and  $\mathcal{P}$ - $\Sigma$ -cyclic if  $A_{(\mathcal{P})}$  is a direct sum of cyclic  $\mathbf{Z}_{(\mathcal{P})}$ -modules. If  $\mathcal{P} = \{p\}$  is a single prime, we drop the parentheses in all of the above notation and terminology, e.g.,  $t_pA = t_{\{p\}}A$ .

There are several standard identities and properties of localizations that we will utilize without comment. For example, if  $A$  is a group, then there is a natural homomorphism  $A \cong A \otimes \mathbf{Z} \rightarrow A_{(\mathcal{P})}$ , whose kernel and cokernel are torsion groups with no  $p$  torsion for any  $p \in \mathcal{P}$ . Another property which we use is the existence of a natural isomorphism  $(fA)_{(\mathcal{P})} \cong fA_{(\mathcal{P})}$ . Finally, if  $\mathcal{P}$  and  $\mathcal{Q}$  are collections of primes,  $\mathcal{P} \cup \mathcal{Q} = \mathcal{R}$ , and  $A$  is a group with finite torsion-free rank, then  $A$  is  $\mathcal{R}$ -free (or  $\mathcal{R}$ - $\Sigma$ -cyclic) iff it is both  $\mathcal{P}$ - and  $\mathcal{Q}$ -free (respectively,  $\mathcal{P}$ - and  $\mathcal{Q}$ - $\Sigma$ -cyclic).

The  $\mathbf{Z}$ -adic completion of a group  $A$  will be denoted by  $LA$ , and if  $p$  is a prime, the  $p$ -adic completion will be denoted by  $L_pA$ . There are natural homomorphisms  $A \rightarrow LA$ ,  $A \rightarrow L_pA$  and  $LA \rightarrow L_pA$ , which determine a natural isomorphism  $LA \cong \prod_p L_pA$ .

#### PRELIMINARIES

If  $A$  and  $B$  are countable, and  $0 \rightarrow Q \rightarrow P \rightarrow A \rightarrow 0$  is a free resolution of  $A$ , then there is a surjection  $\text{Hom}(Q, B) \rightarrow \text{Ext}(A, B)$ . Since  $\text{Hom}(Q, B)$  is the direct product of at most a countable number of copies of  $B$ ,  $\text{Hom}(Q, B)$  has cardinality at most  $c$ , and so the same can be said of  $\text{Ext}(A, B)$ . In many of our arguments, our fundamental strategy is to show that  $\text{Pext}(A, B)$  is non-zero by showing that its cardinality is at least  $c$ . This will, in turn, imply that its cardinality is exactly  $c$ . In other words, though we will primarily be concerned with when  $\text{Pext}(A, B) = 0$ , our arguments actually show that this group either vanishes or it has cardinality  $c$ .

**Lemma 1.** *Suppose  $A$  and  $B$  are countable groups and  $B$  is torsion-free. Let*

$$\mathcal{P} = \{p : pB \neq B\}.$$

*Then  $\text{Pext}(A, B) = 0$  iff  $fA$  is  $\mathcal{P}$ -free.*

*Proof.* If  $X$  is a torsion group, then since  $B$  is torsion-free, we have  $\text{Hom}(X, B) = 0$ . If, in addition,  $X = t_p^*X$ , then  $\text{Ext}(X, B)$  is  $p$ -divisible for every  $p \in \mathcal{P}$  (because multiplication by  $p$  is an automorphism of  $X$ ) and also  $p$ -divisible for every  $p \notin \mathcal{P}$  (since multiplication by  $p$  is an automorphism of  $B$ ). This means that  $\text{Ext}(X, B)$  is divisible; but since  $X$  is torsion, it must also be reduced (see 55.3 of [1]), proving that it is, in fact, zero.

The kernel and cokernel of the homomorphism  $A \rightarrow A_{(\mathcal{P})}$  are torsion groups without  $p$ -torsion for any  $p \in \mathcal{P}$ . It follows from the last paragraph that the homomorphism  $\text{Ext}(A, B) \rightarrow \text{Ext}(A_{(\mathcal{P})}, B)$  is an isomorphism, so that we may assume  $A$  is a  $\mathbf{Z}_{(\mathcal{P})}$ -module. If  $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$  is a short exact sequence, then since multiplication by any prime  $q \notin \mathcal{P}$  is an automorphism of  $A$  and  $B$ , it is also an automorphism of  $X$ . It follows that  $X$  is also a  $\mathbf{Z}_{(\mathcal{P})}$ -module, so that the entire computation can be regarded as taking place over the ring  $\mathbf{Z}_{(\mathcal{P})}$ .

For our next reduction, we begin with the observation that  $\text{Pext}(tA, B) = 0$ . To verify this, suppose  $0 \rightarrow B \rightarrow X \rightarrow tA \rightarrow 0$  is a pure short-exact sequence. Purity implies that  $tX$  maps onto  $tA$ , and since  $B$  is torsion-free, this mapping is also injective, so that  $tX \cong tA$  and the sequence splits. Considering the sequence

$$\text{Pext}(tA, B) \rightarrow \text{Pext}(A, B) \rightarrow \text{Pext}(fA, B) \rightarrow 0,$$

it follows that  $\text{Pext}(A, B) \cong \text{Pext}(fA, B)$ , so that we may assume  $A$  is torsion-free. This implies that  $\text{Ext}(A, B)$  is divisible, so that  $\text{Pext}(A, B) = \text{Ext}(A, B)$ .

To begin the proof, note that if  $A$  is  $\mathcal{P}$ -free, then clearly  $\text{Pext}(A, B) = 0$ . Conversely, if  $A$  is not  $\mathcal{P}$ -free, then by Pontryagin's Theorem, it has a finite-rank pure submodule  $A'$  which fails to be  $\mathcal{P}$ -free. Since there is a surjection  $\text{Pext}(A, B) \rightarrow \text{Pext}(A', B)$ , if we can prove  $\text{Pext}(A', B) \neq 0$ , the result follows. We may assume, therefore, that  $A$  has finite torsion-free rank.

Let  $F$  be a  $\mathcal{P}$ -free subgroup of  $A$  of maximal rank. There is, therefore, an exact sequence

$$\text{Hom}(F, B) \rightarrow \text{Ext}(A/F, B) \rightarrow \text{Ext}(A, B).$$

Observe  $\text{Hom}(F, B)$  is countable so the result will follow if we can show that  $\text{Ext}(A/F, B)$  is uncountable.

Since  $A$  is not  $\mathcal{P}$ -free,  $A/F$  must be infinite. There are two possibilities: either  $A/F$  has a summand isomorphic to  $\mathbf{Z}_{p^\infty}$  for some  $p \in \mathcal{P}$ , or summands isomorphic to  $\mathbf{Z}_{p^{k_p}}$  for an infinite set of primes  $p \in \mathcal{P}$ . In the first case,  $\text{Ext}(A/F, B)$  will have a summand isomorphic to  $\text{Ext}(\mathbf{Z}_{p^\infty}, B) \cong \text{Hom}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_{p^\infty} \otimes B)$  which is uncountable, and in the second, it will have a summand isomorphic to  $\prod \text{Ext}(\mathbf{Z}_{p^{k_p}}, B) \cong \prod B/p^{k_p}B$ , which is also uncountable. In either case, the result has been established.

**Lemma 2.** *Suppose  $A$  and  $B$  are countable groups,  $B$  is reduced and  $p$  is a prime such that  $t_p B$  is unbounded. If  $\text{Pext}(A, B) = 0$ , then  $A$  is  $p$ - $\Sigma$ -cyclic.*

*Proof.* Again, we prove that if  $A$  is not  $p$ - $\Sigma$ -cyclic, then  $\text{Pext}(A, B)$  has cardinality at least  $c$ . By 56.1 of [1] there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A, B_0),$$

so that if  $\text{Pext}(A, B_0)$  has cardinality  $c$ , then so does  $\text{Pext}(A, B)$ . It follows that we may assume that  $B^1 = 0$ .

Since  $0 \rightarrow t_p^* B \rightarrow B \rightarrow f_p^* B \rightarrow 0$  is pure, there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A, f_p^* B),$$

so, once again, we may assume  $B = f_p^* B$ , i.e.,  $t_p^* B = 0$ .

By 57.4 of [1] there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Hom}(A^1, LB/B).$$

Since  $t_p B$  is countable and unbounded, it can be verified that  $t_p(LB/B)$  is a divisible group of cardinality  $c$ . If  $p^\omega t_p A \neq 0$ , we could conclude that  $\text{Hom}(A^1, LB/B)$  has cardinality  $c$ , which implies that  $\text{Pext}(A, B)$  also has cardinality  $c$ . Therefore, we may assume  $p^\omega t_p A = 0$ ; since  $A$  is countable, this implies that  $t_p A$  is  $\Sigma$ -cyclic.

Since  $A_{(p)}$  is not  $\Sigma$ -cyclic (as a  $\mathbf{Z}_{(p)}$ -module), we can conclude that  $fA_{(p)}$  is not  $\Sigma$ -cyclic (since otherwise  $A_{(p)} \cong fA_{(p)} \oplus tA_{(p)}$  would be  $\Sigma$ -cyclic). Hence there is a finite-rank pure subgroup  $A' \subseteq A$  containing  $tA$  such that  $A'_{(p)}$  is not  $\Sigma$ -cyclic. There is a surjection  $\text{Pext}(A, B) \rightarrow \text{Pext}(A', B)$ , so we may assume  $A = A'$  has finite torsion-free rank.

We pause for one further reduction. Let  $N = t_p A \oplus F$  be a  $p$ -basic subgroup of  $A$  (i.e.,  $N$  is  $p$ -pure in  $A$  and  $A/N$  is  $p$ -divisible), where  $F$  is a free subgroup of  $N$ . Note that  $F$  will also be  $p$ -pure in  $A$ . We next let  $M = \{a \in A : \exists m \in \mathbf{Z}, ma \in F, (m, p) = 1\}$ , and  $A' = A/M$ . We need to verify the following elementary facts:

(1)  $M/F$  is a torsion group with  $t_p(M/F) = 0$ . This is obvious from the definition.

(2)  $M$  is pure in  $A$ . If  $a \in M$  and  $ma \in F$  with  $(m, p) = 1$ , then computing  $p$ -heights we have  $ht_M(a) = ht_M(ma) \geq ht_F(ma) = ht_A(ma) = ht_A(a) \geq ht_M(a)$ . If  $q \neq p$ ,  $a \in M$  and  $a = q^k y$  for some  $y \in A$ , then  $y \in M$ , so that  $q$ -heights in  $M$  and  $A$  also agree.

(3)  $tA' = t_p A' \cong t_p A$ , under the natural map  $A \rightarrow A'$ . If  $a \in A$  and  $na \in M$ , then  $n = mp^k$ , where  $(m, p) = 1$ . It follows that  $p^k a \in M$ , so that  $tA' = t_p A'$ . To verify the last isomorphism, since  $M$  is pure in  $A$ ,  $tA$  maps surjectively onto  $tA'$ . Note that  $t_p A \cap M = 0$  and  $t_p^* A \subseteq M$ , so the isomorphism follows.

(4)  $fA'$  is  $p$ -divisible. This follows since  $fA'$  is an epimorphic image of  $A/N$ , which is  $p$ -divisible.

(5)  $fA' \neq 0$ . If  $fA' = 0$ , then  $A \cong M \oplus t_p A$ , so that  $A_{(p)} \cong M_{(p)} \oplus t_p A \cong F_{(p)} \oplus t_p A$ , which is a  $\Sigma$ -cyclic  $\mathbf{Z}_{(p)}$ -module.

The first fact implies  $\text{Hom}(M/F, B) = 0$  so that there is an injection  $\text{Hom}(M, B) \rightarrow \text{Hom}(F, B)$ . Since  $F$  has finite rank and  $B$  is countable, we can conclude  $\text{Hom}(F, B)$  is countable, which implies that  $\text{Hom}(M, B)$  is also countable. Considering the exact sequence

$$\text{Hom}(M, B) \rightarrow \text{Pext}(A', B) \rightarrow \text{Pext}(A, B),$$

we can conclude that if  $\text{Pext}(A', B)$  has cardinality  $c$ , then so does  $\text{Pext}(A, B)$ .

We therefore assume  $A = A'$ , and that  $x \in A$  has infinite order. It follows that if  $n_0, n_1, n_2, \dots$  is the  $p$ -height sequence of  $x$ , then either 1) there is a  $j$  such that  $n_j \geq \omega$ , or 2) there are an infinite number of gaps (i.e., values of  $i$  such that  $n_{i+1} > n_i + 1$ ).

Let  $m_0, m_1, m_2, \dots$  be a strictly increasing sequence of finite ordinals with an infinite number of gaps, such that  $m_i < n_i$  for all but finitely many  $i$ .

Let  $B = \{b_1, b_2, \dots\}$  be an enumeration of  $B$ , and let  $t_p B = \bigoplus_{u \in S} K_u$  where  $S \subseteq \omega$  and for every  $u \in S$ ,  $K_u \neq 0$  is a direct sum of copies of  $\mathbf{Z}_{p^u}$ .

Since  $B^1 = 0$ ,  $B$  embeds as a pure subgroup of  $LB \cong L_p B \oplus \prod_{q \neq p} L_q B$  with  $LB/B$  divisible. We can think of elements of  $L_p B$  as vectors with coordinates in a  $p$ -basic subgroup of  $B$ , which contains the terms  $K_u, u \in S$ .

Choose inductively a strictly increasing sequence  $u_i \in S$ , and elements  $v_i \in K_{u_i}$ , such that

- 1)  $v_i$  generates a summand of  $K_{u_i}$ ,
- 2)  $m_i < u_i$ ,
- 3) for every  $j \leq i$ ,  $b_j$  has  $v_i$ -coordinate divisible by  $p^{m_i}$ .

Let  $\{R_\alpha : \alpha < c\}$  be an almost disjoint family of subsets of  $S$ . For each  $\alpha < c$ , let

$$y_\alpha = \sum_{i \in R_\alpha} p^{m_i - i} v_i \in L_p B \subseteq LB$$

[the fact that  $y_\alpha \in L_p B$  follows from the fact that  $m_0, m_1, m_2, \dots$  has an infinite number of gaps].

Now,  $LB/B$  is divisible and  $x \in A$  has infinite order, so for every  $\alpha < c$ , there is a homomorphism

$$f_\alpha : A \rightarrow LB/B$$

such that  $f_\alpha(x) = y_\alpha + B$ . There is an exact sequence

$$\text{Hom}(A, LB) \rightarrow \text{Hom}(A, LB/B) \rightarrow \text{Pext}(A, B).$$

We claim that the  $f_\alpha$  map to distinct elements of  $\text{Pext}(A, B)$ , proving that this has cardinality  $c$ , as required. If this failed, then for some distinct  $\alpha, \beta < c$ ,  $f_\alpha - f_\beta$  would factor through a homomorphism  $g : A \rightarrow LB$ . Since  $g(x) + B = f_\alpha(x) - f_\beta(x) = y_\alpha - y_\beta + B$ , we can find a  $b_j \in B$  such that  $g(x) = y_\alpha - y_\beta + b_j$ . Choose  $i \geq j$  such that  $i \in R_\alpha - R_\beta$  and  $m_i < n_i$ . Then computing  $p$ -heights, we have

$$n_i = ht(p^i x) \leq ht(p^i g(x)) = ht(p^i y_\alpha - p^i y_\beta + p^i b_j).$$

Notice that the  $v_i$ th coordinate of  $p^i b_j$  is divisible by  $p^{m_i+i} > p^{m_i}$ ,  $p^i y_\beta$  is 0 in this coordinate, and  $p^i y_\alpha$  has  $p^{m_i}$  in this coordinate. It follows that the  $p$ -height of  $p^i y_\alpha - p^i y_\beta + p^i b_j$  is  $m_i$ , but this contradicts  $m_i < n_i$ .

If  $p$  is a prime and  $n \leq \omega$ , then a subgroup  $L$  of  $A$  is  $p^n$ -pure if  $p^m A \cap L = p^m L$  for every  $m \leq n$ . Note that if  $n$  is finite and  $L$  is torsion-free, then this reduces to requiring that  $p^n A \cap L = p^n L$ . We mention the following easy-to-check fact:

**Lemma 3.** *Suppose  $A$  is a group,  $J, K$  are subgroups of  $A$ ,  $j$  and  $k$  are integers with  $jJ \subseteq K$  and  $kK \subseteq J$ ,  $p$  is a prime not dividing  $j, k$  and  $n \leq \omega$ . Then  $J$  is  $p^n$ -pure iff  $K$  is  $p^n$ -pure.*

Let  $\sigma = (n_p : p \text{ is prime})$  be a sequence of finite ordinals. A subgroup  $L$  of  $A$  will be called  $\sigma$ -pure if for each prime  $p$ ,  $L$  is  $p^{n_p}$ -pure in  $A$ . Similarly,  $L$  is almost  $\sigma$ -pure if it is  $p^{n_p}$ -pure for all but finitely many primes  $p$ .

**Corollary 1.** *Given the hypotheses of Lemma 3,  $J$  is almost  $\sigma$ -pure iff  $K$  is almost  $\sigma$ -pure.*

THE MAIN RESULT

**Theorem 1.** *Suppose  $A$  and  $B$  are countable groups and  $B$  is reduced. Let*

$$\begin{aligned} \mathcal{P} &= \{p : pfB \neq fB\}, \\ \mathcal{Q} &= \{p : t_p B \text{ is unbounded}\}. \end{aligned}$$

*In addition, let  $\sigma = (n_p)$ , where  $n_p$  is the exponent of  $t_p B$  if  $t_p B$  is bounded and is 0 whenever  $t_p B$  is unbounded. Then  $\text{Pext}(A, B) = 0$  iff*

- (1)  $fA$  is  $\mathcal{P}$ -free,
- (2)  $A$  is  $\mathcal{Q}$ - $\Sigma$ -cyclic, and
- (3) every finitely generated subgroup of  $A$  is almost  $\sigma$ -pure in  $A$ .

*Proof.* We begin by supposing that (1)–(3) are valid and show that  $\text{Pext}(A, B) = 0$ . By Lemma 1, (1) implies that  $\text{Pext}(A, fB) = 0$ . We now show that (2) and (3) imply that  $\text{Pext}(A, tB) = 0$ , which proves the result by considering

$$\text{Pext}(A, tB) \rightarrow \text{Pext}(A, B) \rightarrow \text{Pext}(A, fB) \rightarrow 0.$$

To this end, we may assume that  $B$  is torsion. Since the kernel and cokernel of the natural map  $A \rightarrow A_{(\mathcal{Q})}$  are torsion groups with no  $q$ -torsion for any  $q \in \mathcal{Q}$ , (2) implies that  $\text{Pext}(A, t_{\mathcal{Q}} B) \cong \text{Pext}(A_{(\mathcal{Q})}, t_{\mathcal{Q}} B) = 0$ , so we need to show  $\text{Pext}(A, t_{\mathcal{Q}}^* B) = 0$ . It follows that we may assume  $B = t_{\mathcal{Q}}^* B$ , i.e., that  $B$  is torsion with bounded  $p$ -components.

Let

$$0 \longrightarrow B \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be a pure sequence. Express  $A$  as an ascending union  $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$  where each  $F_i$  is finitely generated. Our objective is to define a splitting  $\phi : A \rightarrow X$ , defining it inductively, extending it from each  $F_i$  to  $F_{i+1}$ . We will assume one particular property of such a partial splitting  $\phi_i : F_i \rightarrow X$ : If  $p$  is a prime,  $m \leq n_p$  and  $x \in F_i \cap p^m A$ , then  $\phi_i(x) \in p^m X$  (in other words,  $\phi_i$  preserves  $p$ -heights up to  $n_p$ ).

Since  $F_{i+1}$  is almost  $\sigma$ -pure, there is a finite set of primes  $\mathcal{T}$  such that for every  $p \notin \mathcal{T}$ ,  $F_{i+1}$  is  $p^{n_p}$ -pure in  $A$ . We can further expand  $\mathcal{T}$  by adding a finite collection of primes in order to guarantee that  $t_{\mathcal{T}}^*(F_{i+1}/F_i) = 0$ . Let  $B_0 = t_{\mathcal{T}} B$  and  $B_1 = t_{\mathcal{T}}^* B$ , so that  $B = B_0 \oplus B_1$ . Note that  $B_0$  is bounded, so that  $X = B_0 \oplus X_1$ , and there is a pure sequence

$$0 \longrightarrow B_1 \longrightarrow X_1 \xrightarrow{\pi} A \longrightarrow 0.$$

Let  $\rho : X \rightarrow B_0$  and  $\mu : X \rightarrow X_1$  be the usual projections. Note that  $\mu \circ \phi_i : F_i \rightarrow X_1$  is a partial splitting of  $\pi$ . Since  $F_{i+1}$  is finitely generated, by purity there is a partial splitting  $\eta : F_{i+1} \rightarrow X_1$ . It should be observed that we are not assuming  $\eta$  extends  $\phi_i$  or  $\mu \circ \phi_i$ .

There is an exact sequence

$$\text{Hom}(F_{i+1}, B_1) \rightarrow \text{Hom}(F_i, B_1) \rightarrow \text{Ext}(F_{i+1}/F_i, B_1)$$

and since  $B_1 = t_{\mathcal{T}}^* B_1$  and  $t_{\mathcal{T}}^*(F_{i+1}/F_i) = 0$ , the right group vanishes. Next, observe that  $\mu \circ \phi_i - \eta|_{F_i}$  is a homomorphism  $F_i \rightarrow B_1$ , so that  $\mu \circ \phi_i - \eta|_{F_i} = \lambda|_{F_i}$  for some homomorphism  $\lambda : F_{i+1} \rightarrow B_1$ .

The last paragraph shows that  $\mu \circ \phi_i$  extends to a partial splitting  $\gamma = \eta + \lambda : F_{i+1} \rightarrow X_1$ . Note that if  $p \notin \mathcal{T}$ ,  $m \leq n_p$ , and  $x \in p^m A \cap F_{i+1}$ , then by  $p^{n_p}$ -purity,  $x \in p^m F_{i+1}$ , so that  $\gamma(x) \in p^m X_1$ , and hence  $\gamma$  preserves  $p$ -heights up to  $n_p$  for all  $p \notin \mathcal{T}$ .

Next, consider  $\kappa = \rho \circ \phi_i : F_i \rightarrow B_0$ . Let  $n = \prod_{p \in \mathcal{T}} p^{n_p}$ , so that  $nB_0 = 0$ . Note  $\kappa$  induces a homomorphism  $\bar{\kappa} : F_i/(F_i \cap nA) \rightarrow B_0$  and that  $\bar{\kappa}$  does not decrease  $p$ -heights (calculated in  $A/nA$ ) for any  $p \in \mathcal{T}$ . Therefore,  $\bar{\kappa}$  extends to a homomorphism  $A/nA \rightarrow B_0$ . Let  $\kappa'$  be the composition  $F_{i+1} \rightarrow A \rightarrow A/nA \rightarrow B_0$ , so that  $\kappa'$  preserves  $p$ -height up to  $n_p$  for each  $p \in \mathcal{T}$ .

We can now extend  $\phi_i$  to  $\phi_{i+1} : F_{i+1} \rightarrow X$  by setting it equal to  $(\kappa', \gamma) : F_{i+1} \rightarrow B_0 \oplus X_1$ . This is clearly a splitting; to show that it preserves heights, let  $p$  be a prime and  $x \in F_{i+1}$ . If  $p \notin \mathcal{T}$ , then  $\kappa'(x)$  has infinite  $p$ -height, so that the  $p$ -height of  $\phi_{i+1}(x)$  equals the  $p$ -height of  $\gamma(x)$  which is at least the  $p$ -height of  $x \in A$  (whenever this is at most  $n_p$ ). Similarly, if  $p \in \mathcal{T}$ , then the  $p$ -height of  $\phi_{i+1}(x)$  equals the  $p$ -height of  $\kappa'(x)$  which is at least as large as the  $p$ -height of  $x \in A$ . Therefore, one direction of the proof is complete.

We now show that if (1), (2) or (3) fails, then  $\text{Pext}(A, B) \neq 0$ . Suppose first that (1) fails. Lemma 1 implies that  $\text{Pext}(A, fB) \neq 0$ , and since there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A, fB) \rightarrow 0$$

we can conclude  $\text{Pext}(A, B) \neq 0$ .

We next assume (3) fails. Suppose  $F$  is a finitely generated subgroup of  $A$  which is not almost  $\sigma$ -pure. Let  $k$  be an integer such that  $kF$  is free. By Corollary 1,  $kF$  also fails to be almost  $\sigma$ -pure, so there is no loss of generality in assuming that  $F$

is free. Now, let  $\mathcal{S}$  be an infinite set of primes such that  $p^{n_p}A \cap F \neq p^{n_p}F$  for all  $p \in \mathcal{S}$ .

Let  $A'$  be the pure hull of  $F$  (i.e.,  $A'/F = t(A/F)$ ). Since  $A'$  is pure in  $A$ , for each  $p \in \mathcal{S}$ ,  $F$  is not  $p^{n_p}$ -pure in  $A'$ . There is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A', B) \rightarrow 0,$$

so if we can show  $\text{Pext}(A', B) \neq 0$ , we will be able to conclude  $\text{Pext}(A, B) \neq 0$ , as required. We may therefore assume  $A = A'$  has finite rank and  $A/F$  is torsion.

*Claim:*  $\text{Pext}(A, tB)$  has cardinality  $c$ .

Before continuing, we note how this implies  $\text{Pext}(A, B) \neq 0$ . There is an exact sequence

$$\text{Hom}(A, fB) \rightarrow \text{Pext}(A, tB) \rightarrow \text{Pext}(A, B).$$

Note that  $\text{Hom}(A, fB) \cong \text{Hom}(fA, fB)$ , and since  $fA$  has finite rank and  $fB$  has countable rank, this group is countable, and the claim follows.

Therefore, we may assume  $B = tB$  is torsion. For each  $p \in \mathcal{S}$ , let  $\langle r_p \rangle$  be a summand of  $t_p B$  of exponent  $n_p$ . If  $B' = \bigoplus_{p \in \mathcal{S}} \langle r_p \rangle$  and  $\text{Pext}(A, B')$  has cardinality  $c$ , then it follows that  $\text{Pext}(A, B)$  also has cardinality  $c$ , as required. So we may assume  $B = B'$ .

For each  $p \in \mathcal{S}$ , let  $m_p \leq n_p$  be minimal such that  $p^{m_p}A \cap F \neq p^{m_p}F$ , and let  $x_p \in (p^{m_p}A \cap F) - p^{m_p}F$ . The minimality of  $m_p$  implies that  $x_p$  has  $p$ -height  $m_p - 1$  in  $F$  (since otherwise  $x_p \in (p^{m_p-1}A \cap F) - p^{m_p-1}F$ ). Let  $y_p \in F$  satisfy  $p^{m_p-1}y_p = x_p$ . Now choose  $z_p \in A$  such that  $p^{m_p}z_p = x_p$ . The minimality of  $m_p$  also implies that  $p^{m_p-1}z_p \notin F$  (since otherwise  $p^{m_p-1}z_p \in (p^{m_p-1}A \cap F) - p^{m_p-1}F$ ). Let  $A'$  be the subgroup of  $A$  generated by  $F$  together with each  $z_p$  ( $p \in \mathcal{S}$ ). There is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A', B) \rightarrow 0$$

[to verify this, if  $0 \rightarrow Q \rightarrow P \xrightarrow{\rho} A \rightarrow 0$  is a pure sequence with  $P$  and  $Q$   $\Sigma$ -cyclic and  $P' = \pi^{-1}(A')$ , then there are homomorphisms  $\text{Hom}(Q, B) \rightarrow \text{Pext}(A, B) \rightarrow \text{Pext}(A', B)$  whose composite is a surjection]. If we can show the latter group has cardinality  $c$ , it also follows that the first group has that cardinality. So, without loss of generality, we assume  $A = A'$ .

Let  $\{\mathcal{S}_\alpha\}_{\alpha < c}$  be a pairwise almost disjoint family of infinite subsets of  $\mathcal{S}$ . For each  $\alpha < c$ , let  $X_\alpha$  be generated by

- (1.1)  $F$ ,
- (1.2)  $r_p, z_p$ , for  $p \notin \mathcal{S}_\alpha$ ,
- (1.3)  $u_p, v_p$  for  $p \in \mathcal{S}_\alpha$ ,

subject to the relations,

- (2.1)  $p^{n_p}r_p = 0, p^{m_p}z_p = p^{m_p-1}y_p \in F$ , for  $p \notin \mathcal{S}_\alpha$ ,
- (2.2)  $p^{n_p}u_p = p^{n_p}y_p \in F, p^{m_p}v_p = p^{m_p-1}u_p$ , for  $p \in \mathcal{S}_\alpha$ .

These generators and relators determine an extension

$$E_\alpha : 0 \longrightarrow B \xrightarrow{\lambda_\alpha} X_\alpha \xrightarrow{\pi_\alpha} A \longrightarrow 0$$

by assigning for each  $p \in \mathcal{S}_\alpha$ ,

- (3.1)  $\lambda_\alpha(r_p) = u_p - y_p$ ,
- (3.2)  $\pi_\alpha(u_p) = y_p$ ,
- (3.3)  $\pi_\alpha(v_p) = z_p$ .

The sequence  $E_\alpha$  is pure; in fact, for every prime  $p, 0 \rightarrow t_p B \rightarrow t_p X_\alpha \rightarrow t_p A \rightarrow 0$  actually splits. To see this, note that if  $p \notin \mathcal{S}_\alpha$ , then  $t_p X_\alpha = \langle r_p \rangle \oplus \langle pz_p - y_p \rangle$ , and if  $p \in \mathcal{S}_\alpha$ , then  $t_p X_\alpha = \langle u_p - y_p \rangle \oplus \langle pv_p - u_p \rangle$ . In either case, the first summand corresponds to  $t_p B$  and the second to  $t_p A$ .

Let  $p \in \mathcal{S}$ . We claim that  $F$  is  $p^{n_p}$ -pure in  $X_\alpha$  iff  $p \in \mathcal{S}_\alpha$ . To verify this, note that if  $p \notin \mathcal{S}_\alpha$ , then  $p^{m_p-1}y_p \in (p^{m_p}X_\alpha \cap F) - p^{m_p}F$ , so that  $F$  is not  $p^{n_p}$ -pure in  $X_\alpha$ . If  $p \in \mathcal{S}_\alpha$ , then  $F/p^{n_p}F$  can be seen to embed naturally as a summand of  $X_\alpha/p^{n_p}X_\alpha$ , and the claim follows.

We now claim that for distinct  $\alpha, \beta < c$ ,  $E_\alpha$  is not equivalent to  $E_\beta$ , proving that  $\text{Pext}(A, B)$  has the required cardinality. In fact, we show that  $X_\alpha$  and  $X_\beta$  are not isomorphic. To show this is the case, we assume that they are and derive a contradiction. Given such an isomorphism, we can identify  $X_\alpha$  and  $X_\beta$ , giving a single group  $X$ . Note that  $F$  embeds in both  $X_\alpha$  and  $X_\beta$ , but after this identification we may get two distinct subgroups which we denote by  $F_\alpha$  and  $F_\beta$ , respectively. There are integers  $j$  and  $k$  such that  $jF_\alpha \subseteq F_\beta, kF_\beta \subseteq F_\alpha$ . Since  $\mathcal{S}_\alpha - \mathcal{S}_\beta$  is infinite, there is a prime  $p \in \mathcal{S}_\alpha - \mathcal{S}_\beta$  which does not divide  $j, k$ . This means that  $F_\alpha$  is  $p^{n_p}$ -pure in  $A$ , while  $F_\beta$  fails to be  $p^{n_p}$ -pure in  $A$ , contradicting Lemma 3.

We now suppose that it is (2) that fails. If for some  $p \in \mathcal{Q}, A_{(p)}$  fails to be a  $\Sigma$ -cyclic  $\mathbf{Z}_{(p)}$ -module, then by Lemma 2 we can conclude  $\text{Pext}(A, B) \neq 0$ , so we may assume each such  $A_{(p)}$  is  $\Sigma$ -cyclic. In particular, we may assume  $t_{\mathcal{Q}}A = \bigoplus_{p \in \mathcal{Q}} t_p A$  is  $\mathcal{Q}$ - $\Sigma$ -cyclic.

There is a sequence

$$0 \rightarrow t_{\mathcal{Q}}A \rightarrow A_{(\mathcal{Q})} \rightarrow (fA)_{(\mathcal{Q})} \rightarrow 0,$$

so we may assume  $fA$  is not  $\mathcal{Q}$ -free. This implies that there is a pure subgroup  $A' \subseteq A$  of finite torsion-free rank such that  $fA'$  is not  $\mathcal{Q}$ -free. Since there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A', B) \rightarrow 0,$$

if we can show  $\text{Pext}(A', B) \neq 0$ , then  $\text{Pext}(A, B) \neq 0$ , as required. It suffices, then, to assume that  $A = A'$  has finite torsion-free rank.

Let  $F$  be a free subgroup of  $A$  of maximal rank, let  $\mathcal{R}$  be the set of primes  $p \in \mathcal{Q}$  such that  $F$  is  $p^\omega$ -pure in  $A$ , and let  $\mathcal{S} = \mathcal{Q} - \mathcal{R}$  be the remaining ones. Note that  $F_{(\mathcal{R})}$  will be a pure  $\mathbf{Z}_{(\mathcal{R})}$ -submodule of  $A_{(\mathcal{R})}$ . Since  $A_{(\mathcal{R})}/F_{(\mathcal{R})}$  is torsion, it follows that  $A_{(\mathcal{R})}$  splits into  $t_{\mathcal{R}}A \oplus F_{(\mathcal{R})}$ , so that  $(fA)_{(\mathcal{R})}$  is  $\mathcal{R}$ -free. Now for each  $p \in \mathcal{S}$  we already know  $(fA)_{(p)}$  is  $p$ -free, so that if  $\mathcal{S}$  is finite, we can conclude  $(fA)_{(\mathcal{Q})}$  is  $\mathcal{Q}$ -free, contrary to assumption.

It follows that  $\mathcal{S}$  is finite and for each  $p \in \mathcal{S}$  there exists a positive integer  $m_p$  for which  $F$  fails to be  $p^{m_p}$ -pure in  $A$ . For each  $p \in \mathcal{S}$ , let  $t_p B = K_p \oplus L_p$  where  $K_p$  is a bounded subgroup of exponent  $n_p \geq m_p$ . If  $B' = B/(\bigoplus_{p \in \mathcal{S}} L_p)$ , then  $t_{\mathcal{S}}B' \cong \bigoplus_{p \in \mathcal{S}} K_p$ , so that for this  $A$  and  $B'$ , condition (3) fails. By what we've proven already, this implies that  $\text{Pext}(A, B') \neq 0$ , and since there is a surjection

$$\text{Pext}(A, B) \rightarrow \text{Pext}(A, B') \rightarrow 0,$$

we conclude that  $\text{Pext}(A, B) \neq 0$ , as required.

We now point out two consequences of the above proof:

**Corollary 2.** *If  $A$  and  $B$  are countable groups, then  $\text{Pext}(A, B) = 0$  iff  $\text{Pext}(A, fB) = \text{Pext}(A, tB) = 0$ .*

*Proof.* We may clearly assume  $B$  is reduced. Essentially condition (1) is equivalent to  $\text{Pext}(A, fB) = 0$ , and conditions (2) and (3) are equivalent to  $\text{Pext}(A, tB) = 0$ .

**Corollary 3.** *If  $A$  and  $B$  are countable groups, then  $\text{Pext}(A, B)$  is either 0 or it has cardinality  $c$ .*

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849

DEPARTMENT OF MATHEMATICS, WHITMAN COLLEGE, WALLA WALLA, WASHINGTON 99362