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HAUSDORFF DIMENSION AND DOUBLING MEASURES ON METRIC SPACES

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Abstract. Vol′berg and Konyagin have proved that a compact metric space carries a nontrivial doubling measure if and only if it has finite uniform metric dimension. Their construction of doubling measures requires infinitely many adjustments. We give a simpler and more direct construction, and also prove that for any $\alpha > 0$, the doubling measure may be chosen to have full measure on a set of Hausdorff dimension at most α.

Let (X, ρ) be a compact metric space. Vol'berg and Konyagin proved in [VK] that (X, ρ) carries a nontrivial doubling measure μ (there exists $\Lambda \geq 1$ so that $\mu(B(x, 2r)) \leq \Lambda\mu(B(x, r))$ for all $x \in X$ and $r > 0$) if and only if (X, ρ) has finite uniform metric dimension (in each ball $B(x, 2r)$, there exist at most N points with mutual distances at least r). Here $B(x, r) = \{y : \rho(x, y) < r\}$.

Assume that (X, ρ) has finite uniform metric dimension. The construction of doubling measures in [VK] requires infinitely many adjustments which cannot be predicted in advance. In this note, we give a simpler and more direct construction, and prove that given any $\alpha > 0$, there exists a doubling measure on X that has full measure on a set of Hausdorff dimension at most α. Also we observe that a doubling measure may be concentrated on a countable set even when X is a set on the real line of positive length. Some ideas have been adapted from [FKP], [VK] and [T].

1. Theorems and examples

Assume, from now on, that (X, ρ) is a compact metric space of finite uniform metric dimension and that diam $X < 1$.

For each $k \geq 0$, let $S_k = \{x_{k,j} : 1 \leq j \leq J(k)\}$ be a maximal 10^{-k}-net on X (points in S_k having mutual distances at least 10^{-k}, and points outside S_k having distances less than 10^{-k} to S_k), satisfying

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k \subseteq S_{k+1} \subseteq \cdots .$$

Note that S_0 has only one point $x_{0,1}$.

For each $k \geq 0$, let $\{T_{k,j} : 1 \leq j \leq J(k)\}$ be a partition of S_{k+1} satisfying

$$S_{k+1} \cap B(x_{k,j}, 10^{-k}/2) \subseteq T_{k,j} \subseteq S_{k+1} \cap B(x_{k,j}, 10^{-k}). \tag{1.1}$$

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We call elements of $T_{k,j}$ branch points of $x_{k,j}$, the element $x_{k,j}$ an old branch point and the rest new branch points. Since X has finite uniform metric dimension, $T_{k,j}$ has at most N^4 elements.

Let $M \geq N^4$, and let $w_{k,j}$ be weights at $x_{k,j} (k \geq 1)$ so that

$$M^{-1} \leq w_{k,j} \leq 1, \tag{1.2}$$

$$w_{k,j} \equiv w_k \quad \text{on} \quad S_k \backslash S_{k-1}, \tag{1.3}$$

and

$$\sum_{x_{k+1,i} \in T_{k,j}} w_{k+1,i} = 1. \tag{1.4}$$

Theorem 1. *Assume that μ_k $(k \geq 0)$ are measures on X with total mass concentrated on S_k, defined as follows: μ_0 is the unit point measure at $x_{0,1}$; after μ_k is chosen, μ_{k+1} is defined by distributing the mass from $x_{k,j}$ to its branch points in $T_{k,j}$ so that*

$$\mu_{k+1}(\{x_{k+1,i}\}) = w_{k+1,i}\mu_k(\{x_{k,j}\}), \ x_{k+1,i} \in T_{k,j}. \tag{1.5}$$

Then $\{\mu_k\}$ converges in the weak star topology to a doubling measure μ on (X, ρ) with

$$\mu(B(x, 2r)) \leq M^3 N^8 \mu(B(x, r)) \tag{1.6}$$

for each $x \in X$ and $r > 0$.

This construction works because of (1.3)—the weight being a constant at all new branch points in any given generation. This allows us to compare measures of any two nearby branch points, regardless of their ancestors.

When M is large, with a suitable choice of weights, the measure μ is concentrated on a small set. The next theorem extends a result of Tukia [T] on Euclidean space to metric spaces.

Theorem 2. *Given $\alpha > 0$, there exists a doubling measure on (X, ρ) that has full measure on a set of Hausdorff dimension at most α.*

Recall that the β-dimensional Hausdorff content of a set E in X is the number $H_\beta(E) = \inf \sum_j r_j^\beta$, where the infimum is taken over all countable covers of E by balls of radii r_j. The Hausdorff dimension of a set E is $\inf\{\beta : H_\beta(E) = 0\}$.

A doubling measure on a ball in an Euclidean space cannot have full measure on a set of zero Hausdorff dimension. In contrast, the following examples exist for sets having no interiors.

Example 1. *For each $\alpha \in [0, 1]$, there exists a compact set $X \subseteq \mathbb{R}^1$ of Hausdorff dimension α so that every doubling measure on X is purely atomic.*

Example 2. *There exists a compact set $X \subseteq \mathbb{R}^1$ of positive length, so that some doubling measures on X are purely atomic.*

Both examples are essentially in [KW] and were constructed for another purpose.

Let E be the Cantor ternary set on the unit interval, F be the midpoints of all complementary intervals and $X = E \cup F$. Then every doubling measure on X is concentrated on F. A similar construction works for every α in $[0, 1)$. When $\alpha = 1$, we combine an appropriate sequence of such sets together with their limit points.

As for Example 2, let ν be a doubling measure on $\mathbb{R}^1$ having full measure on a set of zero length as constructed in [BA], and let E be a compact subset contained in $[0,1]$ having positive length and zero ν-measure. Let $\mathcal{W}$ be a Whitney decomposition of $(-2,2)\backslash E$, and F be the collection of midpoints of the intervals in $\mathcal{W}$. Let $X = E \cup F \cup \{-2,2\}$, and let μ be the measure on X with total mass on F so that at each $x \in F$, $\mu(\{x\})$ is the ν-measure of the corresponding Whitney interval. Then X and μ have the properties required.

For details, see the examples X and Z in [KW].

2. Proof of Theorem 1

Define history h on $\bigcup_{k\geq 1} S_k$ as follows: $h(x) = (x_{0,1}, x)$ on S_1; and for $x \in T_{k,j} \subseteq S_{k+1}$, $h(x)$ is the $(k+2)$-tuple $(a_0, a_1, \ldots, a_k, x)$, where $(a_0, a_1, \ldots, a_k) = h(x_{k,j})$. We call a_m $(0 \leq m \leq k)$ the m-th generation ancestor of x. These are well-defined because $\{T_{k,j} : 1 \leq j \leq J(k)\}$ is a partition of S_{k+1}.

There is a slight abuse of notation: when $x_{k,j}$ and $x_{\ell,i}$ are the same point in X while considered as branch points in two different generations, $h(x_{k,j})$ and $h(x_{\ell,i})$ have different numbers of components.

For $\ell \geq k+1$, let

$$T_{k,j}^{\ell} = \{x \in S_\ell : \text{ the } \ell\text{th generation ancestor of } x \text{ is } \ x_{k,j}\},$$

and call elements of $T_{k,j}^{\ell}$ the ℓth generation branch points of $x_{k,j}$. Note that $T_{k,j}^{k+1} = T_{k,j}$,

$$T_{k,j}^{\ell} \subseteq T_{k,j}^{\ell+1}, \tag{2.1}$$

and $\{T_{k,j}^{\ell} : 1 \leq j \leq J(k)\}$ is a partition of S_ℓ. Denote by

$$T_{k,j}^{\infty} = \bigcup_{\ell \geq k+1} T_{k,j}^{\ell}$$

all branch points of $x_{k,j}$, and note that

$$T_{k,j}^{\infty} \cap T_{m,i}^{\infty} = \emptyset$$

if neither $x_{k,j}$ nor $x_{m,i}$ is an ancestor of the other.

We claim that for $\ell \geq k+1$,

$$S_\ell \bigcap B(x_{k,j}, 10^{-k}/3) \subseteq T_{k,j}^{\ell} \subseteq S_\ell \bigcap B(x_{k,j}, 10^{-k+1}/9); \tag{2.2}$$

thus

$$\bigcup_{k+1}^{\infty} S_\ell \bigcap B(x_{k,j}, 10^{-k}/3) \subseteq T_{k,j}^{\infty} \subseteq B(x_{k,j}, 10^{-k+1}/9).$$

Therefore, any point in $\bigcup_{k+1}^{\infty} S_\ell$ which is sufficiently close to $x_{k,j}$ is a branch point of $x_{k,j}$, and all branch points of $x_{k,j}$ are not far from $x_{k,j}$. To prove (2.2) let $x \in T_{k,j}^{\ell}$ and follow along its ancestors since $x_{k,j}$; we have $\rho(x_{k,j}, x) < 10^{-k} + 10^{-k-1} + \cdots + 10^{-\ell+1} < 10^{-k+1}/9$; this proves the second inclusion in (2.2). If $x_{\ell,i} \in S_\ell \bigcap B(x_{k,j}, 10^{-k}/3)$, then either $x_{\ell,i} = x_{k+1,p}$ or $x_{\ell,i} \in T_{k+1,p}^{\ell}$ for some p. Apply the second inclusion to $x_{k+1,p}$; we have $\rho(x_{\ell,i}, x_{k+1,p}) < 10^{-k}/9$, and hence $\rho(x_{k+1,p}, x_{k,j}) < 10^{-k}/9 + 10^{-k}/3 < 10^{-k}/2$. In view of (1.1), $x_{k+1,p} \in T_{k,j}$ and hence $x_{\ell,i} \in T_{k,j}^{\ell}$; this proves the first inclusion in (2.2).

The convergence of $\{\mu_k\}$ is now clear.

We note from (1.3), (1.4), (1.5) and (2.1) that for $\ell \geq k+1$,

$$\mu_\ell(T^\ell_{k,j}) = \mu_k(\{x_{k,j}\}), \tag{2.3}$$

and

$$\mu_\ell(\{x_{\ell,i}\}) = \left(\prod_{k+1}^{\ell} w_m\right) \mu_k(\{x_{k,j}\}), \tag{2.4}$$

provided that $x_{\ell,i} \in T^\ell_{k,j}$, and $x_{\ell,i}$ and all ancestors since the $(k+1)$st generation are new branch points.

The main idea of the proof is contained in the following lemma.

Lemma 1. *If $k \geq 1$ and $\rho(x_{k,i}, x_{k,j}) < \frac{2}{9}10^{-k+3}$, then*

$$\mu_k(\{x_{k,i}\})/\mu_k(\{x_{k,j}\}) \leq M^3. \tag{2.5}$$

Proof. For $k=1$, the estimate follows from (1.2) and (1.5). Assume $k \geq 2$ and let $h(x_{k,i}) = (a_0, a_1, \ldots, a_{k-1}, x_{k,i}), h(x_{k,j}) = (b_0, b_1, \ldots, b_{k-1}, x_{k,j})$. Denote by k_0 the largest index for which $a_{k_0} = b_{k_0}$.

If $k_0 < k-3$, we claim that a_m and b_m are new branch points in S_m for each m in $[k_0+2, k-2]$. Otherwise, assume that a_m is an old branch point in S_m; thus a_m and a_{m-1} are the same point in X. Because a_m is an ancestor of $x_{k,i}$, it follows from (2.2) that $\rho(x_{k,i}, a_m) < 10^{-m+1}/9$. Because $a_{m-1} \neq b_{m-1},\ a_{m-1}$ is not an ancestor of $x_{k,j}$; from (2.2) again, we have $\rho(x_{k,j}, a_{m-1}) > 10^{-m+1}/3$. Thus $\rho(x_{k,i}, x_{k,j}) > 10^{-m+1}/3 - 10^{-m+1}/9 > \frac{2}{9}10^{-k+3}$, which is a contradiction. Therefore a_m, and similarly b_m, is a new branch point. In view of (2.4),

$$\mu_{k-2}(\{a_{k-2}\}) = \left(\prod_{k_0+2}^{k-2} w_\ell\right) \mu_{k_0+1}(\{a_{k_0+1}\})$$

and

$$\mu_{k-2}(\{b_{k-2}\}) = \left(\prod_{k_0+2}^{k-2} w_\ell\right) \mu_{k_0+1}(\{b_{k_0+1}\}).$$

As a_{k_0+1} and b_{k_0+1} are branch points of $a_{k_0} = b_{k_0},\ \mu_{k_0+1}(\{a_{k_0+1}\})/\mu_{k_0+1}(\{b_{k_0+1}\}) \leq M$ by (1.2) and (1.5); similarly $M^{-2} \leq \mu_k(\{x_{k,i}\})/\mu_{k-2}(\{a_{k-2}\}) \leq 1$ and $M^{-2} \leq \mu_k(\{x_{k,j}\})/\mu_{k-2}(\{b_{k-2}\}) \leq 1$. From these, (2.5) follows.

If $k_0 \geq k-3$, (2.5) holds because of (1.2) and (1.5). □

Given $x \in X$ and $r > 0$, we shall prove (1.6). Assume that $10^{-k} < r \leq 10^{-k+1}$ for some $k \geq 1$. Because S_{k+1} is a maximal net, $\rho(x, x_{k+1,p}) \leq 10^{-k-1}$ for some p and $T^\infty_{k+1,p} \subseteq B(x_{k+1,p}, 10^{-k}/9) \subseteq B(x, r/4)$. Therefore, by (2.3),

$$\mu(B(x, r/2)) \geq \mu\left(\overline{T^\infty_{k+1,p}}\right) \geq \mu_{k+1}(\{x_{k+1,p}\}). \tag{2.6}$$

Let $\mathcal{J}$ be the set of j's so that $x_{k+1,j} \in B(x, 2r)$; then $\mathcal{J}$ contains at most N^8 elements. We claim that

$$S_\ell \cap B(x, 3r/2) \subseteq \bigcup_{\mathcal{J}} T^\ell_{k+1,j} \quad \text{for each} \quad \ell \geq k+2. \tag{2.7}$$

In fact, given $x_{\ell,i} \in B(x, 3r/2)$, $x_{\ell,i}$ is contained in $T^{\ell}_{k+1,q}$ for some q. Since $T^{\ell}_{k+1,q} \subseteq B(x_{k+1,q}, 10^{-k}/9)$, we have $\rho(x_{k+1,q}, x) \le \rho(x_{k+1,q}, x_{\ell,i}) + \rho(x_{\ell,i}, x) < 10^{-k}/9 + 3r/2 < 2r$. Thus $q \in \mathcal{J}$. This proves (2.7). Therefore

$$\mu_\ell(B(x, 3r/2)) \le \sum_{\mathcal{J}} \mu_\ell(T^{\ell}_{k+1,j}) = \sum_{\mathcal{J}} \mu_{k+1}(\{x_{k+1,j}\})$$

for each $\ell \ge k+2$. Since $\rho(x_{k+1,p}, x_{k+1,j}) \le \rho(x_{k+1,p}, x) + \rho(x, x_{k+1,j}) < 10^{-k-1} + 2r < \frac{2}{9}10^{-k+1}$, we deduce from (2.5) and (2.6) that

$$\mu_\ell(B(x, 3r/2)) \le M^3 N^8 \mu(B(x, r/2).$$

From this, (1.6) follows. And this proves Theorem 1. $\square$

3. Proof of Theorem 2

For $x \in S_k$, recall that $h(x)$ has the form $(x_{0,1}, a_1, a_2, \dots, a_{k-1}, a_k)$ and that the first element $x_{0,1}$ is not a branch point. For $k \ge 1$ and $0 \le p \le k$, denote by

$$S_k(p) = \{x \in S_k : h(x) \quad \text{contains exactly} \quad p \quad \text{old branch points}\}.$$

There are exactly $\binom{k}{p}$ different ways to position p old branch points in $h(x)$; afterwards there are at most $(N-1)^{k-p}$ different ways to place new branch points in the remaining slots. Therefore $S_k(p)$ has at most $\binom{k}{p}(N-1)^{k-p}$ elements. Thus the set

$$\sigma_k(p) = \{x \in S_k : h(x) \ \text{ contains at least } p \ \text{ old branch points}\}$$

has at most $\sum\limits_{m=p}^{k} \binom{k}{m}(N-1)^{k-m}$ elements.

Denoting $\frac{N-1}{M}$ by γ, we prove the following.

Lemma 2. *If $k \ge 1$, then*

$$\mu_k(\sigma_k(p)) \ge \sum_{m=p}^{k} \binom{k}{m}(1-\gamma)^m \gamma^{k-m} \quad \textit{for} \quad 0 \le p \le k. \tag{3.1}$$

Proof. If $k = 1$ and $p = 0$, then $\sigma_1(0) = S_1$ and $\mu_1(\sigma_1(0)) = 1$. If $k = 1$ and $p = 1$, then $\sigma_1(1) = \{\text{the old branch point in } S_1\}$ and $\mu_1(\sigma_1(1)) \ge 1-\gamma$. Hence (3.1) holds for $k = 1$.

Assume that (3.1) is true for some $k \ge 1$. We shall prove the inequality for $k+1$ and all p in $[0, k+1]$. If $p = 0$, then $\mu_{k+1}(\sigma_{k+1}(0)) = 1$. If $p = k+1$, then $\mu_{k+1}(\sigma_{k+1}(k+1)) \ge (1-\gamma)^{k+1}$.

Let $1 \le p \le k$. For $x \in \sigma_{k+1}(p)$, denote by $a_1(x)$ the first generation ancestor of x. Then either $a_1(x)$ is an old branch point and there are at least $p-1$ old branch points in the remaining k slots in $h(x)$, or $a_1(x)$ is a new branch point and there are at least p old branch points in the remaining k slots. From the induction hypothesis, it follows that

$$\begin{aligned}\mu_{k+1}(\sigma_{k+1}(p)) &= \mu_1(\sigma_1(1)) \sum_{m=p-1}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\ &\quad + (1-\mu_1(\sigma_1(1))) \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\ &\geq (1-\gamma) \sum_{m=p-1}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} + \gamma \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\ &= \sum_{n=p}^{k+1} \binom{k+1}{n} (1-\gamma)^n \gamma^{k+1-n}.\end{aligned}$$

The inequality follows from the fact that $\lambda A + (1-\lambda)a \geq (1-\gamma)A + \gamma a$ provided that $\lambda \geq 1-\gamma$ and $A \geq a > 0$. Therefore (3.1) holds for $k+1$. The lemma is proved. $\square$

Assume that M is large enough so that $\gamma = \frac{N-1}{M} < \frac{1}{5}$ and

$$(1-2\gamma)^{-(1-2\gamma)}(2\gamma)^{-2\gamma}(2N)^{2\gamma}10^{-\alpha} < 2^{-\alpha}.$$

Choose p to be $[(1-2\gamma)k]$ in the remaining part of the proof, and let

$$\tau_k = \bigcup \{T_{k,j}^{\infty} : x_{k,j} \in \sigma_k(p)\}.$$

Then for large k,

$$\begin{aligned}H_\alpha(\overline{\tau}_k) &\leq \sum_{m=p}^{k} \binom{k}{m} (N-1)^{k-m} (10^{-k+1})^\alpha \\ &\leq 10k \binom{k}{p} N^{k-p} 10^{-k\alpha} \\ &\leq (1-2\gamma)^{-(1-2\gamma)k-1/2}(2\gamma)^{-2\gamma k-1/2}(2N)^{2\gamma k}10^{-\alpha k} \\ &< 2^{-\alpha k}.\end{aligned} \tag{3.2}$$

The third inequality follows from Stirling's formula ($k! \approx k^{k+1/2}e^{-k}\sqrt{2\pi}$). Note from (3.1) that, for large k,

$$\begin{aligned}\mu(\overline{\tau}_k) &\geq \mu_k(\sigma_k(p)) \\ &= \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\ &= 1 - \sum_{m=0}^{p-1} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\ &> 1 - p\binom{k}{p} (1-\gamma)^p \gamma^{k-p} \\ &> 1 - 10 \left(\frac{e}{4}\right)^{\gamma k}.\end{aligned} \tag{3.3}$$

Here Stirling's formula is again used in the last estimate.

Let

$$\tau = \bigcap_{K \geq 5} \bigcup_{k \geq K} \overline{\tau}_k.$$

It follows from (3.2) and (3.3) that

$$H_\alpha(\tau) = 0 \quad \text{and} \quad \mu(\tau) = 1.$$

This proves Theorem 2. □

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