

## FREE PRODUCTS OF ABELIAN GROUPS IN THE UNIT GROUP OF INTEGRAL GROUP RINGS

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ABSTRACT. We classify finite groups  $G$  which are such that the unit group of the integral group ring  $\mathbf{Z}G$  has a subgroup of finite index which is a non-trivial free product of abelian groups.

For a finite commutative group  $G$  it is well known, and first discovered by Higman, that the unit group  $\mathcal{U}(\mathbf{Z}G)$  of the integral group ring  $\mathbf{Z}G$  is the direct product of the trivial units and a finitely generated free abelian group. In particular, the structure of the unit group is known. For a nonabelian finite group  $G$  the structure of the unit group  $\mathcal{U}(\mathbf{Z}G)$  is much more complicated. The few known structure theorems state that  $\mathcal{U}(\mathbf{Z}G)$  seldom belongs to a well studied class of groups, such as solvable and nilpotent groups (see for example [9, 10]). The reason for this is that  $\mathcal{U}(\mathbf{Z}G)$  contains a free subgroup of rank 2 except when  $G$  is abelian or  $G$  is a Hamiltonian 2-group, a result due to Hartley and Pickel. Recently, Marciniak and Sehgal explicitly described generators for such a free subgroup.

The study of free subgroups in  $\mathcal{U}(\mathbf{Z}G)$  was continued in [4]; it is shown that there are only four finite groups  $G$  which are such that  $\mathcal{U}(\mathbf{Z}G)$  contains a noncyclic free subgroup of finite index. In [5] and [6] a classification is given of the finite groups  $G$  which are such that  $\mathcal{U}(\mathbf{Z}G)$  has a subgroup of finite index which is the direct product of noncyclic free subgroups.

In this paper we answer the following question posed to the authors by Z. Marciniak: describe the finite groups  $G$  which are such that the unit group  $\mathcal{U}(\mathbf{Z}G)$  contains a subgroup of finite index which is a free product of free abelian groups. As in [4, 5, 6] it turns out that such groups can be characterised in terms of the Wedderburn decomposition of their rational group algebra. We will actually answer a slightly more general question.

Throughout the paper the following notation is used. The centraliser of an element  $g$  of a group  $G$  is denoted by  $C_G(g)$ . The dihedral group of order  $2n$  is denoted  $D_{2n}$ , the quaternion group of order  $2n$  is denoted  $Q_{2n}$  and the cyclic group of order  $n$  is denoted  $C_n$ . We also need some additional notation for four particular

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groups of order 16 and one of order 32:

$$\begin{aligned} \mathcal{P} &= \langle a, b \mid a^4 = 1 = b^4, ba = a^3b \rangle, \\ \mathcal{D} &= \langle a, b, c \mid a^2 = b^2 = c^4 = 1, ac = ca, bc = cb, ba = c^2ab \rangle, \\ D_{16}^+ &= \langle a, b \mid a^8 = b^2 = 1, ba = a^5b \rangle, \\ D_8 Y Q_8 &= \text{the central product of } D_8 \text{ and } Q_8, \\ &\quad \text{with their respective centres amalgamated} \end{aligned}$$

By  $M_n(R)$  we denote the  $n$ -by- $n$  matrix ring over a ring  $R$ , and by  $\mathbf{H}^t(K)$  we denote a totally definite quaternion algebra over a real field  $K$ . By  $\mathbf{H}(K)$  we denote the classical quaternion algebra over a field  $K$ . The unit group of a ring  $R$  is denoted by  $\mathcal{U}(R)$ . For a subgroup  $H$  of a finite group  $G$  we denote by  $\widehat{H}$  the idempotent  $\frac{1}{|H|} \sum_{h \in H} h$  of  $\mathbf{Q}G$ .

We are now able to state the theorem.

**Theorem 1.** *Let  $G$  be a finite nonabelian group which is not a Hamiltonian 2-group. Consider the following properties:*

- (1)  $\mathcal{U}(\mathbf{Z}G)$  contains a subgroup of finite index which is a non-trivial free product of abelian groups.
- (2) For any non-torsion unit  $u \in \mathcal{U}(\mathbf{Z}G)$  there exists a positive integer  $n$  such that  $C_{\mathcal{U}(\mathbf{Z}G)}(u^n)$  is abelian-by-finite.
- (3) The simple Wedderburn components of  $\mathbf{Q}G$  are of one of the following six types:  $\mathbf{Q}$ ,  $K$ ,  $\mathbf{H}^t(\mathbf{Q})$ ,  $M_2(\mathbf{Q})$ ,  $M_2(K)$  or  $M_2(\mathbf{H}^t(\mathbf{Q}))$ , where  $K$  is a quadratic imaginary extension of  $\mathbf{Q}$ . Moreover, exactly one matrix component occurs.
- (4) The group  $G$  is isomorphic to one of the following:  $D_6$ ,  $D_8$ ,  $Q_{12}$ ,  $\mathcal{P}$ ,  $\mathcal{D}$ ,  $D_{16}^+$ ,  $Q_8 \times C_3$ ,  $Q_8 \times C_4$  and  $D_8 Y Q_8$ .

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4).

*Proof.* (1)  $\Rightarrow$  (2). Let  $H$  be a subgroup of finite index, say  $n$ , in  $\mathcal{U}(\mathbf{Z}G)$  which is a free product of abelian groups  $H_1, H_2, \dots, H_l$ . Then, for any non-torsion unit  $u$  of  $\mathbf{Z}G$ ,  $1 \neq u^n \in H$ . The description of subgroups of free products, given by Kurosh (see for example [8, Theorem 6.3.1]), implies that  $C_H(u^n)$  is a free product of a free group and conjugates of subgroups of the abelian groups  $H_i$ ,  $1 \leq i \leq l$ . Since the centre of a free product of two nontrivial groups is trivial (see for example [8, Proposition 6.2.6]) and because  $1 \neq u^n$  is in the centre of  $C_H(u^n)$ , it follows that  $C_H(u^n)$  is abelian. Because  $H$  is of finite index in  $\mathcal{U}$ , it is then clear that  $C_H(u^n) = C_{\mathcal{U}(\mathbf{Z}G)}(u^n) \cap H$  is of finite index in  $C_{\mathcal{U}(\mathbf{Z}G)}(u^n)$ . So this shows that (2) holds.

(2)  $\Rightarrow$  (3). Write  $\mathbf{Q}G = \bigoplus M_{n_i}(D_i)$ , where  $m_i \geq 1$  and each  $D_i$  is a division algebra. Let  $\mathcal{O}_i$  be a  $\mathbf{Z}$ -order in  $D_i$ ; then both  $\mathbf{Z}G$  and  $\Lambda = \bigoplus M_{n_i}(\mathcal{O}_i)$  are  $\mathbf{Z}$ -orders in  $\mathbf{Q}G$ . Hence, there exists a positive integer  $n(\Lambda)$  such that  $n(\Lambda)$ -th power of a unit of  $\Lambda$  is a unit in  $\mathbf{Z}G$ . Hence, by [10, Proposition 1.9], periodic subgroups of  $\mathcal{U}(\mathbf{Z}G)$  and of  $\mathcal{U}(M_{n_i}(\mathcal{O}_i))$  are of bounded period. Because such groups are also linear, it follows that periodic subgroups are finite.

If  $e_i$  is the identity of  $M_{n_i}(\mathcal{O}_i)$ , we often identify  $\mathcal{U}(M_{n_i}(\mathcal{O}_i))$  with  $(1 - e_i) + \mathcal{U}(M_{n_i}(\mathcal{O}_i)) \subseteq \mathbf{Q}G$ . So, under this identification,  $\mathbf{Z}G \cap ((1 - e_i) + \mathcal{U}(M_{n_i}(\mathcal{O}_i)))$  will be written as  $\mathbf{Z}G \cap \mathcal{U}(M_{n_i}(\mathcal{O}_i))$ .

First we show that each  $n_i \leq 2$ . Assume the contrary; that is, suppose  $m = n_i \geq 3$ . Let  $E_{ij}$ ,  $1 \leq i, j \leq m$ , be a set of matrix units for  $M_m(\mathcal{O}_m)$ . Then there exists a positive integer  $n(\Lambda)$  such that the group  $H = \langle 1 + n(\Lambda)E_{ij} \mid m \geq j > i > 1 \rangle$

$\subseteq \mathcal{U}(\mathbf{Z}G)$ . Since  $H$  is not abelian and has a non-periodic central element, this yields a contradiction.

Second, we show that there exists precisely one index  $i$  so that  $M_{n_i}(\mathcal{O}_i)$  has infinitely many units, or equivalently the unit group of  $M_{n_i}(\mathcal{O}_i)$  is non-periodic. By the Hartley-Pickel result and its proof (see Theorem 5.1 in [10]), and because  $G$  is not abelian and not a Hamiltonian 2-group, there exists an index  $j$  so that  $\mathcal{U}(M_{n_j}(\mathcal{O}_j)) \cap \mathcal{U}(\mathbf{Z}G)$  contains a non-cyclic free group  $F$ . Suppose now that  $i \neq j$  and  $\mathcal{U}(M_{n_i}(\mathcal{O}_i))$  is infinite. So let  $u \in \mathcal{U}(M_{n_i}(\mathcal{O}_i))$  be a non-periodic unit. Thus, there exists a positive integer  $n(\Lambda)$  so that  $1 + u^{n(\Lambda)} \in \mathcal{U}(M_{n_i}(\mathcal{O}_i)) \cap \mathcal{U}(\mathbf{Z}G)$ , and clearly  $F \subset C_{\mathcal{U}(\mathbf{Z}G)}(u^{n(\Lambda)})$ , in contradiction with (2). This shows the claim.

Next we show that there exists a unique index  $i$  with  $n_i \neq 1$ . Suppose the contrary, that is,  $\mathbf{Q}G$  is a direct sum of division rings and fields. Hence  $G$  is a Hamiltonian group, and since it is not a two group,  $G$  contains  $H = Q_8 \times C_p$  with  $p$  an odd prime. As  $\mathbf{Q}H$  does not contain non-zero nilpotent elements,  $p \geq 7$  (Lemma VI.1.18 in [9]). Hence the Wedderburn decomposition of  $\mathbf{Q}(Q_8 \times C_p)$  contains the direct sum  $\mathbf{Q}(\zeta_p) \oplus \mathbf{H}(\mathbf{Q}(\zeta_p))$ .

Now  $\mathcal{U}(\mathbf{Z}H) \cap \mathcal{U}(\mathbf{Z}[\zeta_p])$  contains a non-periodic element  $u$  and  $\mathcal{U}(\mathbf{Z}H) \cap \mathcal{U}(\mathbf{H}[\zeta_p])$  contains a non-cyclic free group (Hartley- Pickel)  $F_2$ . Clearly  $F_2 \subset C_{\mathcal{U}(\mathbf{Z}G)}(u)$ , again in contradiction with (3).

So far we have shown that there exists a unique index  $j$  with  $n_j = 2$ , and that all other  $n_i$  are 1. Furthermore it follows that for every  $i \neq j$ ,  $\mathcal{U}(\mathcal{O}_i)$  is finite. We now show that also  $\mathcal{U}(\mathcal{O}_j)$  is finite. For this notice that for any  $u \in \mathcal{U}(\mathcal{O}_j) \cap \mathcal{U}(\mathbf{Z}G)$  there exists an integer  $m \geq 2$  such that

$$u_1 = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix},$$

$$u_2 = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$$

and

$$v = \begin{bmatrix} u^m & 0 \\ 0 & u^m \end{bmatrix}$$

belong to  $\mathcal{U}(\mathbf{Z}G)$ ; so the group  $\langle u_1, u_2 \rangle$  is free and  $\langle u_1, u_2 \rangle \subset C_{\mathcal{U}(\mathbf{Z}G)}(v)$ . It follows from the assumption (2) that  $v$ , and thus  $u$ , are periodic. Hence  $\mathcal{U}(\mathcal{O}_i)$  is periodic and thus finite.

The Dirichlet Unit Theorem and Lemma 21.3 in [10] now yield that  $\mathcal{O}_i$  is either the ring of integers of a totally imaginary quadratic extension of  $\mathbf{Q}$ , or an order in a totally definite quaternion algebra over  $\mathbf{Q}$ . In particular, (3) follows.

(3)  $\Rightarrow$  (4) A finite group whose rational group algebra satisfies condition (3) we call *admissible*. Assume  $G$  is admissible. From the group tables of groups of small order (see for example [11]) one can easily deduce that if  $|G| \leq 32$  then  $G$  is one of the groups listed in (4). Hence it is sufficient to establish the following five steps.

- (i) If  $\mathbf{Q}G$  does not have a non-commutative division ring as a simple component, then  $|G| \leq 32$ .
- (ii) If  $\mathbf{Q}G$  has a non-commutative division ring as a simple component, then  $Q_8$  or  $Q_{12}$  is a homomorphic image of  $G$ .
- (iii) If  $Q_{12}$  is a homomorphic image of  $G$ , then  $G = Q_{12}$ .
- (iv) If  $Q_8$  is a homomorphic image of  $G$  and  $|G| > 32$ , then  $G$  is a 2-group and either  $G/\mathcal{Z}(G) \cong C_2 \times C_2$  or  $G$  contains a Hamiltonian subgroup of index 2.

(v) If  $G$  is a 2-group and  $Q_8$  is a homomorphic image of  $G$ , then  $|G| \leq 32$ .

To prove these statements we need six technical lemmas.

**Lemma 2.** *If  $N$  is a subgroup of  $G$ , then  $\mathbf{Q}N$  does not have two matrix rings as simple components, nor does it contain a matrix ring over a division ring with infinitely many units in its orders.*

*Proof.* Since orders in matrix rings have infinitely many units, any matrix component of  $\mathbf{Q}N$  is contained in the matrix component of  $\mathbf{Q}G$ . If  $\mathbf{Q}N$  had two matrix components, then this would yield four orthogonal idempotents in a 2-by-2 matrix ring component of  $\mathbf{Q}G$ , which is impossible.

On the other hand if  $M_2(D_1)$ ,  $D_1$  a division ring, is a simple component of  $\mathbf{Q}H$ , then it follows that  $M_2(D_1) \subseteq M_2(D) = (\mathbf{Q}G)e$ , where  $e$  is the primitive central idempotent defining the matrix component of  $\mathbf{Q}G$ . Since, by assumption, orders in  $D$  have a finite unit group, the same holds for orders in  $D_1$ .  $\square$

**Lemma 3.** *The possible orders of elements of  $G$  are 1, 2, 3, 4, 6, 8 or 12, and the centre of  $G$  has exponent at most 6.*

*Proof.* Let  $g \in G$  be an element of order  $n$ . Consider the group algebra  $\mathbf{Q}\langle g \rangle = \mathbf{Q}(\zeta_n) \oplus M$  with  $\mathbf{Q}(\zeta_n) = \mathbf{Q}\langle g \rangle f$ ,  $\zeta_n$  a primitive  $n$ -th root of unity,  $f$  a primitive idempotent and  $M$  a direct sum of fields. Let  $e$  be a primitive central idempotent of  $\mathbf{Q}G$  such that  $fe \neq 0$ . So  $\mathbf{Q}(\zeta_n) \subseteq (\mathbf{Q}G)e$ .

From the assumptions we obtain that

$$[(\mathbf{Q}G)e : \mathbf{Q}] = [(\mathbf{Q}G)e : \mathcal{Z}((\mathbf{Q}G)e)] [\mathcal{Z}((\mathbf{Q}G)e) : \mathbf{Q}] = 2^i,$$

with  $0 \leq i \leq 4$ , and therefore  $\varphi(n) = 2^i$ ,  $0 \leq i \leq 4$ . Note that actually  $i < 4$ , for otherwise  $(\mathbf{Q}\langle g \rangle)e = (\mathbf{Q}G)e$  is a field of  $\mathbf{Q}$ -dimension 16, a contradiction. Clearly we obtain that  $n = 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24$  or  $30$ .

In the cases  $n = 10, 15, 16, 20, 24$  or  $30$ , one can write

$$\mathbf{Q}\langle g \rangle = \mathbf{Q}\langle g \rangle f_1 \oplus \mathbf{Q}\langle g \rangle f_2 \oplus M',$$

where  $M'$  is a direct sum of fields, and  $F_1 = \mathbf{Q}\langle g \rangle f_1$  and  $F_2 = \mathbf{Q}\langle g \rangle f_2$  are fields each of which have ring of integers with infinite group of units. Since orders in the division ring components of  $\mathbf{Q}G$  have finite unit groups, it follows that  $f_1e \neq 0$ ,  $f_2e \neq 0$  and  $\mathbf{Q}Ge = M_2(D)$ ,  $D$  a division ring. Hence  $F_i \subseteq ef_i\mathbf{Q}Gef_i \cong D$ . Thus  $D = F_i$ , and thus an order in  $D$  contains infinitely many units, again a contradiction. So we have shown that the order of  $g$  is 1, 2, 3, 4, 5, 6, 8 or 12. From the above method it also follows that the centre of  $G$  has exponent at most 6.

So it remains to show that  $n \neq 5$ . Assume the contrary. Then  $\mathbf{Q}\langle g \rangle = \mathbf{Q}\langle g \rangle \hat{g} \oplus \langle g \rangle \mathbf{Q}(1 - \hat{g})$ ,  $\mathbf{Q}\langle g \rangle \hat{g} \cong \mathbf{Q}$  and  $\mathbf{Q}\langle g \rangle (1 - \hat{g}) \cong \mathbf{Q}(\zeta_5)$ . Since the unit group of  $\mathbf{Z}[\zeta_5]$  is infinite and because  $G$  is admissible,  $(1 - \hat{g})e \neq 0$ ,  $(\mathbf{Q}G)e$  is a matrix ring and  $(1 - \hat{g})f = 0$  for any primitive central idempotent  $f \in \mathbf{Q}G$  with  $f \neq e$ . Hence  $1 - \hat{g} = (1 - \hat{g})e$ . Therefore the idempotent  $1 - \hat{g}$  belongs to  $\mathbf{Q}Ge$ . Now,  $1 - \hat{g}$  is central, for otherwise  $\mathbf{Q}\hat{g}(1 - \hat{g}) \subseteq (1 - \hat{g})e\mathbf{Q}G(1 - \hat{g}) \cong D$ . So  $D \cong \mathbf{Q}(\zeta_5)$  and therefore an order in  $D$  has infinitely many units, a contradiction. Hence it follows that  $1 - \hat{g} = e$ . Since  $e$  is central this implies that the group  $\langle g \rangle$  is normal in  $G$ . Now, since  $(\mathbf{Q}G)e$  is nonabelian, there exists  $h \in G$  such that  $he \notin \langle g \rangle e$ . Because  $g$  has order 5 and the order of  $h$  is 2, 3, 4, 5, 6, 8 or 12, we may assume, by replacing  $h$  by some power if necessary, that  $he$  has order 2, 3 or 5. Furthermore, there must exist such an  $h$  with order of  $he$  different from 5; for otherwise  $Ge$  is a 5-group and thus

$\mathbf{Q}G$  would not have a simple component which is a 2-by-2 matrix ring. If  $he$  has order 2, then, since  $G$  does not have elements of order 10,  $\langle ge, he \rangle \cong D_{10}$ . However, because of Lemma 2, this is impossible as  $\mathbf{Q}D_{10}$  has  $M_2(F)$  as a simple component, where  $F$  a quadratic real field extension of  $\mathbf{Q}$ . If  $he$  has order 3, then  $\langle ge, he \rangle$  is a group of order 15, and thus commutative. Again this yields a contradiction, as  $G$  does not have elements of order 15.  $\square$

**Lemma 4.** *If  $N$  is a subgroup of  $G$  and  $\mathbf{Q}N$  has a simple component which is a matrix ring (that is,  $N$  is nonabelian and not Hamiltonian), then  $N$  is admissible. Furthermore, if  $\mathbf{Q}N$  has a matrix component of the same  $\mathbf{Q}$ -dimension as the matrix component of  $\mathbf{Q}G$ , then  $N = G$ .*

*Proof.* Let  $f$  be a primitive central idempotent of  $\mathbf{Q}N$  and let  $e$  be a primitive central idempotent of  $\mathbf{Q}G$  with  $ef \neq 0$ . Since  $(\mathbf{Q}N)f$  is simple,  $(\mathbf{Q}N)f \subseteq (\mathbf{Q}G)e$ .

We first deal with the case that  $(\mathbf{Q}N)f$  is a matrix ring. Note that, since  $\mathbf{Q}G$  has only one matrix component,  $e$  is the only primitive central idempotent of  $\mathbf{Q}G$  with  $ef \neq 0$ . Hence  $ef = f$ . First assume  $(\mathbf{Q}N)fe = (\mathbf{Q}G)e = M_2(D)$ , where  $D$  is a division ring (this occurs precisely when  $\dim_{\mathbf{Q}}((\mathbf{Q}N)fe) = \dim_{\mathbf{Q}}((\mathbf{Q}G)e)$ ). Then  $f = fe = e$ . Hence  $Gf = Ge \subseteq \mathbf{Q}Nf$ . Consequently,  $G \subseteq \text{supp}(Gf) \subseteq \text{supp}((\mathbf{Q}N)f) \subseteq N$ ; and thus  $G = N$ .

Next assume  $(\mathbf{Q}N)f$  is properly contained in  $(\mathbf{Q}G)e$  and  $(\mathbf{Q}N)f$  is a matrix ring. Then  $\dim_{\mathbf{Q}}((\mathbf{Q}N)f) = 4$  or  $8$ . So either  $(\mathbf{Q}N)f = M_2(\mathbf{Q})$  or  $(\mathbf{Q}N)f = M_2(F)$ , where  $F$  is a quadratic field extension of  $\mathbf{Q}$ . Now, if  $(\mathbf{Q}N)f = M_2(F)$ , then a set of matrix units of  $(\mathbf{Q}N)f$  is also a set of matrix units of  $(\mathbf{Q}G)e$ ; and thus the centraliser of these matrix units in  $(\mathbf{Q}N)f$  is contained its centraliser in  $(\mathbf{Q}G)e$ . Thus  $F \subset D$ , where  $D$  is a division ring such that  $(\mathbf{Q}G)e = M_2(D)$ . So the ring of integers  $\mathcal{O}$  of  $F$  is contained in an order of  $D$ . The latter has finitely many units; therefore the same holds for  $\mathcal{O}$ . Hence  $F$  is quadratic imaginary. So the matrix component of  $\mathbf{Q}N$  is of a good type.

Next assume  $(\mathbf{Q}N)f = D$ , a division ring. Note that the simple matrix component of  $\mathbf{Q}N$  and  $D$  cannot belong to the 2-by-2 matrix component of  $\mathbf{Q}G$ . So, up to isomorphism,  $(\mathbf{Q}N)f = D$  is contained in a division ring component of  $\mathbf{Q}G$ . Since an order in the latter has finitely many units, the same holds for an order in  $D$ . Hence  $\mathbf{Q}Nf = \mathbf{Q}Ge$  is a totally definite quaternion algebra over  $\mathbf{Q}$ ; that is, it is again of a good type. It follows that  $N$  is admissible.  $\square$

Note that the rational group algebra of the Hamiltonian group  $Q_8 \times C_3$  has a matrix ring as simple component. It follows from the two previous lemmas that the only Hamiltonian subgroups or homomorphic images of  $G$  which are such that their rational group algebra is a sum of division rings are Hamiltonian 2-groups. We will use this remark several times.

**Lemma 5.** *If  $N$  is a subgroup of  $G$  such that  $\mathbf{Q}N$  has a matrix ring and a non-commutative division ring as simple components, then  $N = G$ .*

*Proof.* Let  $e_1$  and  $e_2$  be central primitive idempotents of  $\mathbf{Q}N$  such that  $\mathbf{Q}Ne_1$  is a matrix ring and  $\mathbf{Q}Ne_2$  is a non-commutative division ring. Once again, it is clear that  $(\mathbf{Q}N)e_1$  and  $(\mathbf{Q}N)e_2$  are not contained in the same matrix simple component of  $\mathbf{Q}G$ . Since noncommutative division ring components of  $\mathbf{Q}G$  have  $\mathbf{Q}$ -dimension 4,  $\mathbf{Q}Ne_2 = \mathbf{Q}Ge$  is a division ring, for some primitive central idempotent  $e$  of  $\mathbf{Q}G$ . Hence  $G \subseteq \text{supp}(Ge_2) \subset \mathbf{Q}Ne_2 \subseteq N$ , and thus  $G = N$ .  $\square$

**Lemma 6.** *Assume  $N$  is a nonabelian subgroup of  $G$  which is not a Hamiltonian 2-group. If  $|N| \geq 32$ , then  $N = G$ .*

*Proof.* Since  $N$  is nonabelian,  $\dim_{\mathbf{Q}} \mathbf{Q}N(1 - \widehat{N}') \geq |N|/2 \geq 16$ . We consider two cases. First assume  $\mathbf{Q}N(1 - \widehat{N}')$  is simple. In particular,  $\mathbf{Q}N$  has only one noncommutative simple component. If this component were a division ring, then  $N$  would be a Hamiltonian 2-group, in contradiction with the assumption. So, by Lemma 4,  $\mathbf{Q}N(1 - \widehat{N}') = M_2(D)$ ,  $D$  a division ring of dimension at most 4 over  $\mathbf{Q}$ . Because  $\dim_{\mathbf{Q}} \mathbf{Q}N(1 - \widehat{N}') \geq 16$ , we obtain  $\dim_{\mathbf{Q}} D = 4$ . Hence, by Lemma 4,  $N = G$ .

Second, assume  $\mathbf{Q}N(1 - \widehat{N}')$  is not simple; then it must contain a matrix ring and a non-commutative division ring. So, by Lemma 5,  $N = G$ .  $\square$

**Lemma 7.** *If  $N$  is a proper subgroup of  $G$ , then  $N$  is abelian or a Hamiltonian 2-group, or  $|N| \leq 16$ .*

*Proof.* By Lemma 5,  $\mathbf{Q}N$  does not contain a matrix ring and a non-commutative division ring as simple components. So either  $\mathbf{Q}N$  is a sum of division rings, and then  $N$  is abelian or a Hamiltonian 2-group; or  $\mathbf{Q}N$  has only one non-commutative component and this component is a matrix ring. By Lemma 4, this matrix ring is of the type  $M_2(F)$  with  $\mathbf{Q}$ -dimension of  $F$  equal to 1 or 2. Then

$$8 \geq \dim_{\mathbf{Q}} \mathbf{Q}N(1 - \widehat{N}') \geq N/2.$$

So  $|N| \leq 16$ .  $\square$

We now proceed with the proofs of the five earlier mentioned main steps.

*Proof of (i).* Suppose  $G$  does not have a noncommutative division ring as simple component. Since  $G$  is admissible, this means that  $\mathbf{Q}G(1 - \widehat{G})$  is a matrix ring. Hence  $\dim_{\mathbf{Q}} \mathbf{Q}G(1 - \widehat{G}) \leq 16$ . Now, once again,  $\mathbf{Q}G \cong \mathbf{Q}[G/G'] \oplus \mathbf{Q}G(1 - \widehat{G}')$ . So it follows that  $|G| \leq \frac{|G|}{2} + 16$ . Hence  $|G| \leq 32$ .  $\square$

*Proof of (ii).* Assume  $(\mathbf{Q}G)e$  is a noncommutative division ring component of  $\mathbf{Q}G$ , where  $e$  is a primitive central idempotent. Because of Lemma 3, the only possible primes dividing  $|G|$  are 2 and 3, and  $G$  does not contain elements of order 16 nor of order 9. Hence, since  $Ge$  is a subgroup of a division ring (and thus is fixed point free), it follows from [7, Theorem 18.1] that the 3-Sylow subgroups of  $Ge$  are cyclic, and thus of order 3, and the 2-Sylow subgroups are either cyclic or a generalised quaternion group. Hence they are either  $C_2$ ,  $C_4$ ,  $C_8$ ,  $Q_8$  or  $Q_{16}$ . However, because of Lemma 4, the group  $Q_{16}$  is excluded. Hence  $|Ge|$  is 24, 12 or 8 (note that  $Ge$  is nonabelian and cannot be of order 6). Since the group algebra  $\mathbf{Q}[Ge]$  has at most one matrix ring as a simple component, it follows from the Wedderburn decompositions of the groups of such orders that: if  $|Ge| = 8$ , then  $Ge = Q_8$ ; if  $|Ge| = 12$ , then  $Ge = Q_{12}$ ; if  $|Ge| = 24$ , then  $Ge = Q_8 \times C_3$ . However,  $\mathbf{H}(\mathbf{Q})$  is the only noncommutative division ring simple component of the rational group algebra of the latter group. Since  $Q_8 \times C_3$  is not embedded in  $\mathbf{H}(\mathbf{Q})$ , this situation actually does not occur.  $\square$

*Proof of (iii).* Let  $N$  be a normal subgroup of  $G$  with  $G/N \cong Q_{12}$ . Clearly  $G$  contains normal subgroups  $H_1$  and  $H_2$  so that  $N \subseteq H_1 \subseteq H_2$ ,  $G/H_1 \cong D_6$  and  $H_2/H_1$  is of order 3. By Lemma 3,  $G$  does not contain elements of order 9; hence there exists an element  $g \in G$  of order 3 such that  $H_2/H_1$  is generated by the natural image of  $g$  in this group.

We claim that  $\langle g \rangle$  is a normal subgroup of  $G$ . Indeed, note that the subgroup  $\langle g, H_1 \rangle$  is normal in  $G$  and the idempotent  $\widehat{g}(1 - \widehat{H}_1)$  belongs to  $(\mathbf{Q}G)(1 - \widehat{H}_1)$ , a direct sum of division rings (note that  $\mathbf{Q}G\widehat{H}_1$  contains a matrix ring as simple component). Hence,  $\widehat{g}\widehat{H}_1$  and  $\widehat{g}(1 - \widehat{H}_1)$  are central in  $\mathbf{Q}G$ . Hence,  $\widehat{g}$  is central and thus  $\langle g \rangle$  is normal in  $G$ .

Again by Lemma 3, there exists  $h \in G$  such that the order of  $h$  is  $2^i$ ,  $1 \leq i \leq 3$ , and the natural image of  $h$  in  $G/H_1$  is of order 2. Then, since  $\langle g \rangle$  is normal in  $G$ ,  $T = \langle g, h \rangle = \{h^i g^j\}$ ; and similarly as above we obtain that  $\widehat{T} = \widehat{g}\widehat{h}$  is central in  $\mathbf{Q}G$ . Therefore  $T$  is a normal nonabelian subgroup of  $G$  with  $|T| = 2^i 3$ . Thus  $|T| = 24, 12$  or  $6$ .

If  $|T| = 6$  then  $T \cong D_6$ . It follows that  $G \cong T \times H_1 \cong D_6 \times H_1$ . Hence  $\mathbf{Q}G$  has more than one simple component which is a matrix ring, a contradiction.

If  $|T| = 12$ , then since  $D_6$  is a homomorphic image of  $T$ , the subgroup  $T$  is admissible. Hence  $T = Q_{12}$ , and thus, by Lemma 5,  $T = G$ .

If  $|T| = 24$ , once again as before one shows that  $T$  is admissible. Thus  $T = C_3 \times Q_8$ , and thus it follows again that  $T = G$ . However, this is impossible, as  $Q_{12}$  is not a homomorphic image of  $C_3 \times Q_8$ .

So we have shown that  $G = Q_{12}$ . □

*Proof of (iv).* Assume  $G/N \cong Q_8$  and  $|G| > 32$ . Because of Lemma 7, any subgroup containing  $N$  and of index 2 in  $G$  is either abelian or a Hamiltonian 2-group. In the latter case  $G$  itself is a 2-group and we are done. So we may assume that all such subgroups are abelian. Then it follows easily that  $G/\mathcal{Z}(G) \cong C_2 \times C_2$ . In particular  $G$  is nilpotent, and thus  $G$  is the direct product of its Sylow 2 and 3 subgroups, which we denote by  $G_2$  and  $G_3$  respectively. Because of Lemma 3,  $G_3 \subseteq N$  is an elementary abelian 3-group. Note that  $G_2$  is nonabelian. If  $\mathbf{Q}G_2$  contains a matrix ring as a simple component, then clearly  $G_3$  is trivial, as  $G$  is admissible. So  $G_2$  is a Hamiltonian 2-group. We claim that  $G_3$  is trivial. For otherwise,  $Q_8 \times C_3$  is a subgroup of  $G$ . Since  $\mathbf{Q}[Q_8 \times C_3]$  contains a matrix ring and a noncommutative division ring as simple components, it follows from Lemma 5 that  $G = Q_8 \times C_3$ , a contradiction as  $|G| > 32$ . □

*Proof of (v).* Assume  $G$  is a 2-group with normal subgroup  $N$  such that  $G/N = Q_8$ . We have to prove that  $|G| \leq 32$ . Suppose the contrary. By step (iv), either  $G/\mathcal{Z}(G) \cong C_2 \times C_2$  or  $G$  contains a Hamiltonian normal subgroup  $N$  of index 2.

First we deal with the case that  $G/\mathcal{Z}(G) \cong C_2 \times C_2$ . Note that then  $G' = \{1, c\}$ , with  $c$  a central element of order 2, and that  $\mathcal{Z}(G)$  has exponent 2 or 4 (by Lemma 3). So  $(\mathbf{Q}G)^{\frac{1-c}{2}}$  is the direct sum of all the noncommutative simple components of  $\mathbf{Q}G$ , and all these simple components are generalised quaternion algebras over  $\mathbf{Q}$  or  $\mathbf{Q}(i)$ . In the latter case such a simple component is a matrix ring. Hence, since  $G$  is admissible, there exists at most one matrix component over  $\mathbf{Q}(i)$ . We claim that  $\mathcal{Z}(G)$  is an elementary abelian 2-group. Suppose the contrary. Since  $|G| > 32$ ,  $|\mathcal{Z}(G)| > 8$  and thus  $\mathcal{Z}(G)$  contains  $C_4 \times C_2$ . It follows that  $\mathbf{Q}[C_4 \times C_2]^{\frac{1-c}{2}}$  is contained in the centre of  $(\mathbf{Q}G)^{\frac{1-c}{2}}$  and contains 2 copies of the field  $\mathbf{Q}(i)$ , a contradiction. So indeed  $\mathcal{Z}(G)$  has exponent 2, and therefore there exist elements  $a, b \in G$  such that both  $a$  and  $b$  have order 4 and  $Q_8$  is an epimorphic image of  $\langle a, b \rangle$ . Since  $c \in \langle a, b \rangle$  and because  $c$  has order 2, the group  $\langle a, b \rangle$  is normal in  $G$  and is of order at most 32. Because  $\mathcal{Z}(G)$  is of exponent 2, it follows that  $G \cong \langle a, b \rangle \times E$ , where  $E$  is an elementary abelian 2-group. As  $G$  is admissible, the

group algebra  $\mathbf{Q}\langle a, b \rangle$  contains a matrix ring as a simple component. Furthermore, because  $\mathbf{Q}G$  has precisely one matrix ring as simple component, we obtain that  $E$  is trivial. This finishes the proof for this type of groups.

Next assume  $N$  is a Hamiltonian subgroup of index 2 in  $G$ . Therefore  $Q_8 = \langle i, j \rangle \subseteq N$  and  $N = Q_8 \times E$ , with  $E$  an elementary abelian 2-group. Note that  $i^2$  is central in  $G$  as  $N$  is normal in  $G$  and  $i^2$  is the unique nontrivial commutator in  $N$ . Let  $a \in G$  be such that its natural image in  $G/N$  generates this group. We claim that we may assume that  $Q_8$  is normal in  $G$ . This is clear if the cyclic groups  $\langle i \rangle$  and  $\langle j \rangle$  are normal in  $G$ . On the other hand, if, say,  $\langle i \rangle$  is not normal, then take for  $Q_8$  the group  $\langle i, aia^{-1} \rangle$ . Next we claim that  $E$  is not central. For otherwise, one can write  $G = \langle Q_8, a \rangle \times T$ , for some subgroup  $T$  of  $E$ . Since  $G$  is admissible, this implies that  $T$  is trivial, and thus  $|G| \leq 32$ , a contradiction.

Let  $c_1 \in E$  be non-central in  $G$ . Because  $\frac{1-i^2}{2}$  is a central idempotent, one of the idempotents  $\left(\frac{1-i^2}{2}\right)\left(\frac{1-c_1}{2}\right)$  or  $f = \left(\frac{1+i^2}{2}\right)\left(\frac{1-c_1}{2}\right)$  is central. It is easily seen that the former one does not commute with  $a$ . Hence  $f = afa^{-1} = \left(\frac{1+i^2}{2}\right)\left(\frac{1-c_2}{2}\right)$ , where  $c_2 = ac_1a^{-1} \neq c_1$ . Consequently,  $c_1 = i^2c_2$ .

Write  $a^2 = qe$  with  $q \in Q_8$  and  $e \in E$ . Then,  $qe = aa^2a^{-1} = aqea^{-1} \in Q_8aea^{-1}$ . So,  $e(aea^{-1})$  is an element of order at most 2 in  $Q_8$ , and thus either  $ea = ea$  (that is,  $e$  is central in  $G$ ) or  $aea^{-1} = i^2e$ .

Assume  $e$  is central. Then  $f_2 = \frac{1+e}{2}$  and  $f_3 = \frac{1-e}{2}$  are orthogonal central idempotents. So one of the idempotents  $f_2\frac{1+c_1}{2}$ ,  $f_3\frac{1+c_1}{2}$  has to be central. Because  $ac_1a^{-1} = i^2c_1$ , this is only possible if  $e = 1$ . But then the group  $\langle Q_8, a, c_1 \rangle$  has order 32. Since its group algebra has non-central idempotents, this group is admissible (by Lemma 4). Since not all its elements of order 2 are central, this group is not Hamiltonian. Hence it must be the group  $D_8 Y Q_8$ . Lemma 4 then yields that  $G = D_8 Y Q_8$ .

Next assume  $aea^{-1} = i^2e$ . Then the group algebra of the group  $H = \langle Q_8, a, e \rangle$  has  $\frac{1+e}{2}$  as a non-central idempotent. So  $\mathbf{Q}H$  contains a matrix component, and thus, by Lemma 4,  $H$  is admissible. Since  $|H| = 32$  and because  $H$  has noncentral elements of order 2, we obtain that  $H = D_8 Y Q_8$ . Hence, by Lemma 4,  $H = G$ .  $\square$

As mentioned earlier, from the group tables of groups of small order one obtains that (4)  $\Rightarrow$  (3). Since the groups listed in (4) only have one simple component with infinitely many units, and because the centraliser of a non-central element of  $GL_2(\mathbf{C})$  is abelian, one obtains that (4)  $\Rightarrow$  (2), except possibly for  $D_8 Y Q_8$ .

Finally we remark that finite groups with reduced  $\mathbf{Q}$ -degrees either 1 or a fixed number  $k$  have been investigated in [2, 3]. Among the several results obtained, it is shown that such groups are metabelian. If furthermore these groups have a faithful irreducible representation, then a classification of these groups can be deduced from the methods used in [2, 3]. Note that subgroups of  $M_2(D)$ ,  $D$  a division ring, also have been classified by Banieqbal in [1]. The use of these papers could potentially simplify some of our proofs. However, we elected to present a comprehensive proof.

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