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COHOMOLOGY OF CERTAIN CONGRUENCE SUBGROUPS OF THE MODULAR GROUP

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Abstract. In this note we compute the integral cohomology groups of the subgroups $\Gamma_0(n)$ of $SL(2, \mathbf{Z})$ and the corresponding subgroups $P\Gamma_0(n)$ of its quotient, the classical modular group, $PSL(2, \mathbf{Z})$.

1. Introduction

Let $\Gamma_0(n)$ denote the subgroup of $SL(2, \mathbf{Z})$ consisting of matrices whose lower left entry is divisible by the integer $n \geq 2$ and let $P\Gamma_0(n) = \Gamma_0(n)/(\pm I)$. It is well-known (see [7], p. 11) that the group $PSL(2, \mathbf{Z})$ is the free product of a group of order 2 and a group of order 3. The Kurosh subgroup theorem then tells us that $P\Gamma_0(n)$ is the free product of finitely many copies of $\mathbf{Z}$, $\mathbf{Z}/2$, and $\mathbf{Z}/3$. Hence the integral cohomology of $P\Gamma_0(n)$ is free abelian in dimension one, trivial for higher odd dimensions, and sums of copies of $\mathbf{Z}/2$ and $\mathbf{Z}/3$ in positive even dimensions. Determining the number of summands of each type in the cohomology is equivalent to determining the number of summands of each type in the free product (see [7], p. 127). These were computed for arbitrary n in [2], but that computation is inaccurate. (In fact, the numbers given in [2] fail in general to be integers; the difficulties appear to begin with problems in counting the number of torsion summands.) The rational Euler characteristic of these groups was computed in [4], while in [1] their integral cohomology was computed for n prime. In [3], the cohomology was computed for n prime or a product of two distinct primes. In this paper we shall give a quick, easy computation of the integral cohomology of these groups for all n. To state our result, we define functions $a(n)$, $b(n)$, and $c(n)$ as follows. Let $a(n)$ be the index of $\Gamma_0(n)$ in $SL(2, \mathbf{Z})$. Let $b(n)$ be the number of roots of $x^2 + x + 1$ $(mod\ n)$, and let $c(n)$ be the number of roots of $x^2 + 1$ $(mod\ n)$. Finally, define

$$r(n) = 1 + \frac{a(n) - 4b(n)}{6} - \frac{c(n)}{2}.$$

We have the following two theorems.

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Theorem 1.1. *The cohomology of* $P\Gamma_0(n)$ *is given by the formula*

$$H^i(P\Gamma_0(n),\mathbf{Z}) = \begin{cases} \mathbf{Z}, & \text{if } i=0; \\ \mathbf{Z}^{r(n)}, & \text{if } i=1; \\ (\mathbf{Z}/3)^{b(n)} \oplus (\mathbf{Z}/2)^{c(n)}, & \text{if } i \geq 2 \text{ is even;} \\ 0, & \text{for other values of } i. \end{cases}$$

Theorem 1.2. *The cohomology of* $\Gamma_0(n)$ *is given by the formula*

$$H^i(\Gamma_0(n),\mathbf{Z}) = \begin{cases} \mathbf{Z}, & \text{if } i=0; \\ \mathbf{Z}^{r(n)}, & \text{if } i=1; \\ (\mathbf{Z}/2)^{r(n)}, & \text{if } i \geq 3 \text{ is odd;} \\ (\mathbf{Z}/3)^{b(n)} \oplus (\mathbf{Z}/2)^{c(n)-1} \oplus \mathbf{Z}/4, & \text{if } i \geq 2 \text{ is even and } c(n) > 0; \\ \mathbf{Z}/2 \oplus (\mathbf{Z}/3)^{b(n)}. & \text{if } i \geq 2 \text{ is even and } c(n) = 0. \end{cases}$$

The functions $a(n)$, $b(n)$, and $c(n)$ are well-known. We have $a(n) = n\prod_{p|n}(\frac{p+1}{p})$; $b(n) = 2^k$ if $n = 3^{l_0}p_1^{l_1}\dots p_k^{l_k}$ is the prime factorization of n, where $l_0 \leq 1$ and all $p_i \equiv 1$ $(mod\ 6)$, and is zero otherwise; and $c(n) = 2^k$ if $n = 2^{l_0}p_1^{l_1}\dots p_k^{l_k}$, where $l_0 \leq 1$ and all $p_i \equiv 1$ $(mod\ 4)$, and is zero otherwise.

We shall conclude this paper with some easy applications of our results to the cohomology of the groups $PSL(\mathbf{Z}[1/n], 2)$.

2. Preliminaries

Henceforth, let G denote the group $SL(2,\mathbf{Z})$, and let K denote the subgroup $\Gamma_0(n)$ for some fixed n. Let PK denote $P\Gamma_0(n)$. We shall begin by describing an action of G on the cubic tree. (Recall that the cubic tree is the tree whose vertices all have order three.) It is well-known that G acts on the cubic tree; cf. [7]. Our goal here is to give a description that facilitates the calculation of the action of K. We define X' to be the subset of the integer lattice in the plane given by

$$X' = \{\begin{pmatrix} w \\ y \end{pmatrix} \mid (w,y) = 1\},$$

and define X to be the quotient of X' by the action of $\pm I$. Then the action of G on the integer lattice preserves X, and $\pm I$ are the only elements of G that fix two distinct points of X. We identify, by abuse of notation, X with the subset of X' consisting of points $\binom{w}{y}$ with $w \geq 1$, together with the point $\pm\binom{0}{1}$. Let Δ be the set of triangles whose vertices are elements

$$\begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}, \text{ and} \begin{pmatrix} t \\ u \end{pmatrix}$$

for which $t = w + x$, $u = y + z$, and $wz - xy = \pm 1$. We shall represent such a triangle in matrix form,

$$T = \begin{pmatrix} w & x & t \\ y & z & u \end{pmatrix}.$$

Without loss of generality, we may require that

$$\det \begin{pmatrix} w & x \\ y & z \end{pmatrix} = 1.$$

Note that the triangles of this form provide a triangulation of a subset of the right half-plane.

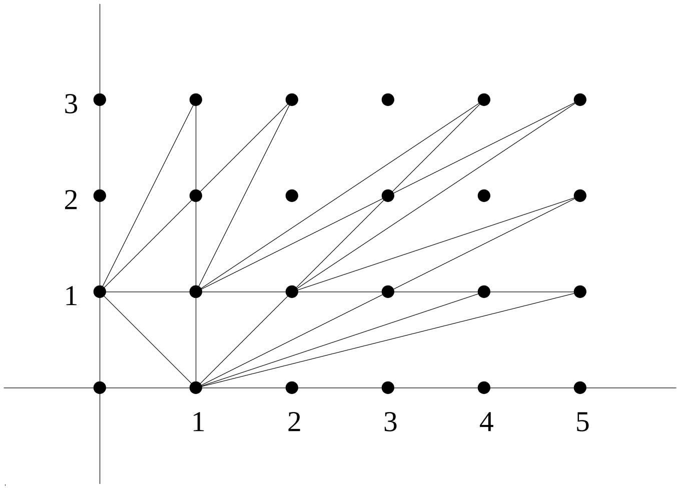

FIGURE 1. Some of the triangles in the first quadrant

The action of G on X induces an action of G on Δ. For T as above, we define three associated elements of G by

$$T_1 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \quad T_2 = \begin{pmatrix} t & -w \\ u & -y \end{pmatrix}, \quad T_3 = \begin{pmatrix} x & -t \\ z & -u \end{pmatrix}.$$

Now consider the action of K on Δ. We observe that if at least one of y, z, u is relatively prime to n, then T is equivalent under the action of K to a triangle of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k+1 \end{pmatrix}$ for some $0 \le k \le n-1$. We further note that if a triangle $S \in \Delta$ is K-equivalent to T, then for some $1 \le i \le 3$, $T_i S^{-1} \in K$.

We conclude that under the projection $G/K \to \Delta/K$, the inverse image of the class of an element T of Δ/K consists of three distinct elements of G/K, except in the case for which $T_2 T_1^{-1}$ (and hence $T_3 T_1^{-1}$) is in K. We call triangles in the first case triangles of Type 1 and in the second case, triangles of Type 2. The triangle T is of Type 2 if and only if $y^2 + yz + z^2 \equiv 0 \ (mod\ n)$. Also in this case, T is K-equivalent to a unique triangle of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k+1 \end{pmatrix}$ for some $0 \le k \le n-1$.

As an example, take $n = 3$, and in Figure 1 consider the triangle with vertices $\binom{2}{1}$, $\binom{3}{1}$, and $\binom{5}{2}$, which we represent in our notation as

$$T = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix},$$

which is a representative of the triangle

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

We observe that the matrix

$$S_2S_1^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

is an element of K of order 6 that preserves the triangle S.

From the above discussion, we conclude

Lemma 2.1. *The cardinality of the set X/K is given by*

$$\operatorname{card}(X/K) = \frac{a(n) + 2b(n)}{3}.$$

We may now consider the dual graph to this complex: The set of vertices of this graph is Δ, and two vertices have an edge in common provided the corresponding triangles are adjacent. This graph is clearly the cubic tree. To compute the cohomology of K, we shall appeal to the following theorem of Serre [7].

Theorem 2.2. *Let a group Γ act without inversion on a tree Y. Let Σ_0 (resp. Σ_1) denote a system of representatives of the vertices (resp. the edges) of Y, and for each vertex x (resp. edge y) let Γ_x (resp. Γ_y) be its stabilizer in Γ. For each Γ-module M, one has an exact cohomology sequence*

$$\dots H^i(\Gamma, M) \to \prod_{x\in\Sigma_0} H^i(\Gamma_x, M) \to \prod_{y\in\Sigma_1} H^i(\Gamma_y, M) \to H^{i+1}(\Gamma, M) \to \dots .$$

The connecting maps are induced by the respective inclusions of groups.

Here, to act without inversion means that there is no element of Γ that exchanges two adjacent vertices of Y. So, to apply this theorem, we shall alter our tree by adding a new vertex at the midpoint of each edge that is inverted by an element of K. If an edge of the tree is inverted by K, the two triangles corresponding to the ends of this edge have two of their vertices in common which are exchanged by an element of K. Suppose these two vertices are $\binom{w}{y}$ and $\binom{x}{z}$. Then the only matrices in G that interchange them are $\pm\begin{pmatrix} wy + xz & -(w^2 + x^2) \\ y^2 + z^2 & -(wy + xz) \end{pmatrix}$. If these matrices are elements of K, then $y^2 + z^2$ must be divisible by n; hence both y and z must be relatively prime to n. By the remarks above, there is a unique pair of triangles of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k+1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ k-1 & 1 & k \end{pmatrix}$ for $1 \le k \le n$ that are exchanged by K. Hence the number of K-equivalence classes of edges that are inverted is $c(n)$.

We see that by adding vertices as described above to bisect each edge inverted by K, we have added $c(n)$ K-equivalence classes of vertices. We call vertices of this type vertices of Type 3, and the vertices corresponding to triangles of Types 1 and 2, vertices of Types 1 and 2, respectively. Thus the total number of equivalence classes of vertices is given by

$$v(n) = \frac{a(n) + 2b(n)}{3} + c(n).$$

To obtain the number $e(n)$ of K-equivalence classes of edges, we do some elementary counting to get

$$e(n) = \frac{a(n) + c(n)}{2}.$$

3. Cohomology calculations

Under the action of K (resp. PK), the stabilizer of vertices of Type 1 is $\mathbf{Z}/2$ (resp. $\{0\}$), the stabilizer of vertices of Type 2 is $\mathbf{Z}/6$ (resp. $\mathbf{Z}/3$), and that of vertices of Type 3 is $\mathbf{Z}/4$ (resp. $\mathbf{Z}/2$). The cohomology of cyclic groups is given by

$$H^i(\mathbf{Z}/n;\mathbf{Z}) = \begin{cases} \mathbf{Z}, & \text{if } i=0; \\ 0, & \text{if } i \text{ is odd}; \\ \mathbf{Z}/n, & \text{otherwise.} \end{cases}$$

The initial portion of the cohomology sequence

$$0 \to H^0(\Gamma, M) \to \prod_{x\in\Sigma_0} H^0(\Gamma_x, M) \to \prod_{y\in\Sigma_1} H^0(\Gamma_y, M) \to H^1(\Gamma, M) \to 0\ .$$

is

$$0 \to \mathbf{Z} \to (\mathbf{Z})^{v(n)} \to (\mathbf{Z})^{e(n)} \to H^1(\Gamma) \to 0,$$

for both $\Gamma = K$ and $\Gamma = PK$. In higher degrees, we have

$$0 \to H^{2i}(\Gamma, M) \to \prod_{x\in\Sigma_0} H^{2i}(\Gamma_x, M) \to \prod_{y\in\Sigma_1} H^{2i}(\Gamma_y, M) \to H^{2i+1}(\Gamma, M) \to 0\ .$$

For $\Gamma = PK$, the fourth term in this sequence is trivial, so the second and third are isomorphic and the fifth is also trivial. For $\Gamma = K$, the sequence is

$$\begin{aligned} 0 \to H^{2i}(K) &\to (\mathbf{Z}/2)^{v(n)-c(n)} \oplus (\mathbf{Z}/3)^{b(n)} \oplus (\mathbf{Z}/4)^{c(n)} \\ &\to (\mathbf{Z}/2)^{e(n)} \to H^{2i+1}(K) \to 0\ . \end{aligned}$$

From this we easily obtain the values stated in Theorems 1.1 and 1.2.

4. Applications

Let n be an integer and let $p_1, \dots, p_k$ be the distinct prime factors of n. There is a spectral sequence (see [6], p. 95, and [4], pp. 813–816) converging to $H^*(PSL(2,\mathbf{Z}[1/n]))$ whose E_1-term is given by

$$E_1^{s,t} = \bigoplus H^t(P\Gamma_0(p_{i_1}\dots p_{i_s}))^{2k-s},$$

where the summation is over all s-element subsets of $\{p_1,\dots,p_k\}$. In particular, for $s=0$, $E_1^{0,*}$ is the sum of 2^k copies of the cohomology of $PSL(2,\mathbf{Z})$.

From the existence of this spectral sequence, we may use the results of this paper to obtain immediately the following results, which were computed inductively in [3].

Proposition 4.1. *In dimension one we have* $H^1(PSL(2,\mathbf{Z}[1/n])) = \{0\}$.

Proposition 4.2. *For* $m \geq k+2$, $H^m(PSL(2,\mathbf{Z}[1/n]))$ *is a finite abelian group possessing only 2- and 3-torsion.*

We remark that for $t \neq 1$ it is fairly easy to compute d_1 and thus obtain $E_2^{s,t}$. To state the results of this computation, for n as above and $j = 0,1,2$ we define the functions

$$\beta_j(n) = \mathrm{card}\{p_i \mid 1 \leq i \leq k \text{ and } b(p_i) = j\}$$

and

$$\gamma_j(n) = \mathrm{card}\{p_i \mid 1 \leq i \leq k \text{ and } c(p_i) = j\}.$$

We have

Proposition 4.3. *For $t \geq 2$, t even,*

$$E_2^{s,t} = (\mathbf{Z}/3)^{2^{\beta_0(n)\binom{\beta_2(n)}{s}}} \oplus (\mathbf{Z}/2)^{2^{\gamma_0(n)\binom{\gamma_2(n)}{s}}}.$$

Furthermore, $E_2^{0,0} = \mathbf{Z}$, and $E_2^{s,0} = 0$ for $s > 0$.

This permits us to bound the abelianization of $PSL(2, \mathbf{Z}[1/n])$.

Corollary 4.4. *The abelianization $H_1(PSL(2, \mathbf{Z}[1/n]))$ of $PSL(2, \mathbf{Z}[1/n])$ is a subgroup of*

$$(\mathbf{Z}/3)^{2^{\beta_0(n)}} \oplus (\mathbf{Z}/2)^{2^{\gamma_0(n)}}.$$

We have no idea how to compute higher differentials or how to solve the extension problem at E_∞, so we shall not pursue this further at present. We note, however, that in dimensions greater than $k+1$, the integral cohomology coincides with the Farrell cohomology, which has been computed by N. Naffah in [5]. Naffah's results show that for $m \geq k+2$,

$$H^m(PSL(2, \mathbf{Z}[1/n])) \simeq \bigoplus_{s=0}^{k} E_2^{s,m-s},$$

so in this range, $E_\infty^{s,m-s} = E_2^{s,m-s}$ and the extensions are trivial.

Since in rational cohomology the spectral sequence is concentrated on the line $t = 1$, the only differential is d_1 and $H^{s+1}(SL(2, \mathbf{Z}[1/n]), \mathbf{Q}) \simeq E_2^{s,1} \otimes \mathbf{Q}$. Although the computation of this differential is beyond the scope of this note, we plan to return to it in future work. From E_1, we get the following crude upper bounds.

Proposition 4.5. *The rational cohomology $H^*(SL(2, \mathbf{Z}[1/n]), \mathbf{Q})$ is trivial in dimensions greater than $k+1$, and for $2 \leq s \leq k+1$ we have*

$$\operatorname{rank}(H^s(SL(2, \mathbf{Z}[1/n]), \mathbf{Q})) \leq 2^{k+1-s} \sum r(p_{i_1} \dots p_{i_s}),$$

where the summation is over all s-element subsets of $\{p_1, \dots, p_k\}$.

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